The Izergin-Korepin model

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Résumé : Parmi les modèles de mécanique statistique classique avec interaction les systèmes intégrables de Yang–Baxter (YB) jouent un rôle particulier. Le modèle central dans la théorie des systèmes intégrables YB est le modèle à six vertex. Plusieurs méthodes ont été développées pour étudier le modèle à six vertex. Notre but est de comprendre la physique du modèle à dix-neuf vertex d'Izergin–Korepin (IK), qui peut être vu comme une généralisation du modèle à six vertex. On donne une vue d'ensemble de l'Ansatz algébrique de Bethe pour le modèle IK basé sur la matrice R à dix-neuf vertex et on propose une nouvelle présentation pour les états propres de la matrice de transfert associée. On adresse aussi la question du calcul des produits scalaires pour le modèle IK. Un objet important dans la théorie des produits scalaires est la fonction de partition avec des conditions aux bords de domaine. Pour cette fonction de partition, définie pour le modèle IK, on obtient une relation de récurrence pour laquelle on trouve la solution dans un cas particulier. La théorie de la représentation du groupe quantique ($U_q(A_2^{(2)})$) associé au modèle IK nous permet d'obtenir toutes les représentations de dimension plus élevée pertinentes pour ce modèle (les modules de Kirillov–Reshetikhin (KR)). Ceci est réalisé dans la présentation de Drinfeld des groupes quantiques. Cette présentation a des avantages techniques quand on calcule les matrices R par la formule de Khoroshkin–Tolstoy (KT). On l'utilise pour calculer la matrice R evaluée sur le produit tensoriel de la représentation fondamentale et d'un module KR de dimension plus élevée. D'un autre côté, la présentation de Drinfeld montre la connexion entre les sous-algèbres de Borel du groupe quantique $U_q(A_2^{(2)})$ et les algèbres d'oscillateurs q-deformés (Osc_q). Ces algèbres sont étroitement liées à la définition (par la théorie de la représentation) d'un certain type de matrices de transfert : les opérateurs Q; ces opérateurs jouent un rôle central dans la théorie des relations fonctionnelles des modèles intégrables. On utilise les algèbres de type Osc_q dans la formule KT pour calculer quelques matrices L, qui sont utilisées pour construire les opérateurs Q. Finalement, on considère un cas particulier de l'état fondamental du modèle IK avec paramètre de deformation q égal à une racine de l'unité. Dans ce cas, on calcule explicitement les valeurs propres de différentes matrices de transfert, y compris de l'opérateur Q. On utilise ce dernier résultat pour obtenir l'état fondamental du modèle IK pour des petites tailles.

Mots clés : Modèles intégrables, modèle d'Izergin–Korepin, Ansatz algébrique de Bethe, fonction de partition avec des conditions aux bords de domaine, théorie de la représentation du groupe quantique $U_q(A_2^{(2)})$, opérateurs Q, formule de Khoroshkin–Tolstoy

Subject : The Izergin-Korepin Model

Abstract: Among the models of interacting classical statistical mechanics the Yang–Baxter (YB) integrable systems play a special role. The central model in the theory of YB integrable systems is the six vertex model. Many powerful techniques were developed to study the six vertex model. The model under consideration is the Izergin–Korepin (IK) nineteen vertex model, which can be viewed as a generalization of the six vertex model. Our aim is to understand the physics of the IK model using the extensions of the methods which were applied to the six vertex model. We review the algebraic Bethe Ansatz for the IK model based on the nineteen-vertex R-matrix and propose a new presentation for the eigenstate of the relevant transfer matrix. We also address the question of the calculation of the scalar products of the IK model. An important object in the theory of scalar products is the domain wall boundary partition function. For this partition function defined for the IK model we derive a recurrence relation and solve it in a special case. We move on to the representation theory of the underlying quantum group ($U_q(A_2^{(2)})$), for which we compute all higher dimensional irreducible representations which are relevant for the IK model (Kirillov–Reshetikhin (KR) modules). The latter is accomplished in the so-called Drinfeld presentation of quantum groups. This presentation has technical advantages for computations of the R-matrices by means of the Khoroshkin–Tolstoy (KT) formula. We use this to compute the R-matrix in a tensor product of the fundamental representation and a generic higher dimensional KR module. On the other hand, the Drinfeld presentation makes apparent the connection between the Borel subalgebras of the quantum group $U_q(A_2^{(2)})$ and the q-deformed oscillator algebras (Osc_q). The latter algebras are closely related to the representation theoretic definition of special transfer matrices: the Q-operators; these operators are central in the theory of functional relations of integrable models. We use the Osc_q type algebras in the KT formula to compute some L-matrices which are used to build the Q-operators. Finally, we consider a special case of the ground state of the IK model when the deformation parameter q is equal to a root of unity. In this case we compute explicitly the ground state eigenvalues of various transfer matrices including the Q-operator. We use the latter result to compute the components of the ground state of the IK model for small systems.

Keywords: Integrable models, Izergin–Korepin model, algebraic Bethe Ansatz, domain wall boundary partition function, representation theory of the quantum group $U_q(A_2^{(2)})$, Q-operators, Khoroshkin–Tolstoy formula

Contents

Introduction

In this thesis we study a quantum system of interacting spins on a line subject to a special symmetry. The model of interest was introduced by Izergin and Korepin [67], thus we refer to it as the Izergin–Korepin model. Quantum models are normally defined by their Hamiltonians H possessing the required symmetry. For the Izergin–Korepin model we denote the Hamiltonian by H_{IK}. We ask the standard question: *what are the physical characteristics of the system under consideration*? In order to answer that one needs to solve several problems. The first is the problem of diagonalizing the Hamiltonian. If we denote the eigenvectors of our Hamiltonian H by $|\psi\rangle$ and the eigenvalues by E, then we want to solve the eigenvalue problem

$$H|\psi\rangle = E|\psi\rangle. \tag{0.0.1}$$

Secondly, we would like to know the physical observables. These are encoded in the corresponding operators, denote one by $\mathcal{O}$. The second problem is to compute the expectation value

$$\langle\mathcal{O}\rangle_T = \frac{Tr\left(\mathcal{O}e^{-H/T}\right)}{Tr\left(e^{-H/T}\right)}. \tag{0.0.2}$$

The temperature T is kept finite here. If we put it equal to 0, then the calculation is reduced to finding the expectation value of $\mathcal{O}$ with respect to the lowest energy state vectors $|\psi_0\rangle$.

$$\langle\mathcal{O}\rangle_{T=0} = \frac{\langle\psi_0|\mathcal{O}|\psi_0\rangle}{\langle\psi_0|\psi_0\rangle}. \tag{0.0.3}$$

Solving these problems allows us to understand the physical characteristics of the system defined by the Hamiltonian H. However, calculations of this kind are very difficult in general. The cases of noninteracting systems are usually tractable, however, for realistic systems interactions are essential. There exist classes of interacting models witch retain the solvability property. A very important class of these models is called the Yang–Baxter quantum integrable models. The Izergin–Korepin model is a member of this class. To rephrase the first sentence, this thesis addresses the problem of solving Eqs. (0.0.1) and (0.0.2) for the physical system which is described by the Izergin–Korepin Hamiltonian $H = H_{IK}$. The Yang–Baxter solvability of the Izergin–Korepin model provides us with a number of tools, some of which we use in order to understand how to solve (0.0.1) and (0.0.2). We restrict ourselves to the study of the finite size systems, thus we will not address numerous interesting and important topics associated to the thermodynamic limit.

In Section 0.1 we give the Hamiltonian H_{IK} and switch to an equivalent description of the Izergin–Korepin (IK) model in terms of the transfer matrices of a classical two dimensional statistical model: the IK nineteen vertex model. We proceed with discussing various approaches to study integrable models and outline a strategy for the case of the IK model. The motivation for the study of the IK model is convenient to present at the end of the introduction. The outline of the thesis closes this chapter.

0.1 The transfer matrix method

We start our discussion from the Bullough–Dodd model[1] [35]. This is a classical model that describes a scalar field in two dimensions

$$\partial_+\partial_- F = -m^2 \left(e^{-2F} - e^F\right). \tag{0.1.1}$$

It is an important nonlinear differential equation from a theoretical point of view as well as for applications. We are interested in the quantum version of this model. This was obtained by Izergin and Korepin [67] by the quantization of the inverse scattering problem for the above equation. In the inverse scattering problem the scattering data is expressed in terms of the monodromy matrix. The so-called classical r-matrix plays a special role here. It tells us how the elements of the monodromy matrix Poisson-commute. In the quantized version, the commutation of the monodromy matrices is obtained with the aid of the quantum R-matrix. The trace over the monodromy matrix is called the transfer matrix $T(\lambda)$, where lambda is called the spectral parameter. The R-matrix-commutation of the monodromy matrices leads to

$$[T(t_1),T(t_2)] = 0, \tag{0.1.2}$$

Thus the eigenvectors of $T(t_i)$ are independent of t_i. What we called before the Izergin–Korepin model is the one dimensional quantum model defined by the Hamiltonian

$$H = T^{-1}(t)\frac{dT(t)}{dt}|_{t=0}. \tag{0.1.3}$$

Therefore, the Hamiltonian is a term in the expansion of the transfer matrix with respect to the parameter t and the diagonalization of the Hamiltonian follows from the diagonalization of the transfer matrix. The seemingly more difficult problem of diagonalizing the transfer matrices can, in fact, be handled more efficiently by means of the quantum inverse scattering method. Therefore, instead of working with the Hamiltonian we will be mostly focused on the transfer matrix approach to the problem. Before turning to the discussion of the problem in this setting we give the explicit form of the Hamiltonian.

1. This model was introduced by Tzitzeika [119] in the context of differential geometry and later reintroduced by Bullough and Dodd [35], and then studied by Zhiber and Shabat [125] and Mikhailov [99]. One can find this same model referred to as the model of a combination of several names from the list of mentioned authors.

Consider the N-fold tensor product $V \otimes .. \otimes V$ where V is a finite dimensional vector space. This is the physical space $\mathcal{H}$. Set $V = \mathbb{C}^3$ with the basis v_+, v_0 and v_-. The Hamiltonian can be represented as a matrix acting on the vectors in $\mathcal{H}$

$$H = \sum_{j=1}^{N} H_{j,j+1}. \tag{0.1.4}$$

Where each operator $H_{j,k}$ acts on the j-th and k-th term in $\mathcal{H}$, $H_{j,k} \in \mathrm{End}(V \otimes V)$. We assume also periodic boundary conditions, so $H_{N,N+1}$ makes sense upon the identification $H_{N,N+1} = H_{N,1}$. Denote the matrix units by $\epsilon_{i,j}$, then each term in the summation in (0.1.4) can be written in the basis of Gell-Mann matrices

$$\begin{aligned}
\lambda_1 &= \epsilon_{1,3} + \epsilon_{3,1}, & \lambda_2 &= i(\epsilon_{1,3} - \epsilon_{3,1}),\\
\lambda_3 &= \epsilon_{1,1} - \epsilon_{3,3}, & \lambda_4 &= \epsilon_{1,2} + \epsilon_{2,1},\\
\lambda_5 &= i(-\epsilon_{1,2} + \epsilon_{2,1}), & \lambda_6 &= \epsilon_{2,3} + \epsilon_{3,2},\\
\lambda_7 &= i(\epsilon_{2,3} - \epsilon_{3,2}), & \lambda_8 &= 3^{-1/2}(\epsilon_{1,1} - 2\epsilon_{2,2} + \epsilon_{3,3}),
\end{aligned}$$

as follows [2]

$$\begin{aligned}
H_{j,k} = {} & (q^{1/2} + q^{-1/2})(q^2 + q^{-2})(\lambda_1 \otimes \lambda_1 + \lambda_2 \otimes \lambda_2)\\
& + i(q^{1/2} + q^{-1/2})(q^2 - q^{-2})(-\lambda_1 \otimes \lambda_2 + \lambda_2 \otimes \lambda_1)\\
& + 2(q^{1/2} + q^{-1/2})\lambda_3 \otimes \lambda_3\\
& + (q^{3/2} + q^{-3/2})(q + q^{-1})(\lambda_4 \otimes \lambda_4 + \lambda_5 \otimes \lambda_5 + \lambda_6 \otimes \lambda_6 + \lambda_7 \otimes \lambda_7)\\
& + i(q^{3/2} + q^{-3/2})(q - q^{-1})(\lambda_4 \otimes \lambda_5 - \lambda_5 \otimes \lambda_4 + \lambda_6 \otimes \lambda_7 - \lambda_7 \otimes \lambda_6)\\
& + (q - q^{-1})^2(\lambda_4 \otimes \lambda_6 + \lambda_6 \otimes \lambda_4 - \lambda_5 \otimes \lambda_7 - \lambda_7 \otimes \lambda_5)\\
& + i(q^2 - q^{-2})(-\lambda_4 \otimes \lambda_7 + \lambda_7 \otimes \lambda_4 - \lambda_5 \otimes \lambda_6 + \lambda_6 \otimes \lambda_5)\\
& + \frac{2}{3}(-(q^{1/2} + q^{-1/2}) + 2(q^{3/2} + q^{-5/2}) + 2(q^{5/2} + q^{-5/2}))\lambda_8 \otimes \lambda_8\\
& + 3^{-3/2}(-(q^{1/2} + q^{-1/2}) + 2(q^{3/2} + q^{-3/2}) - (q^{5/2} + q^{-5/2}))(\lambda_8 \otimes \mathrm{id} + \mathrm{id} \otimes \lambda_8).
\end{aligned}$$

0.2 Algebraic Bethe Ansatz

First of all, we must note that we restrict ourselves within a subclass of methods which are used to study integrable models. We do not mention many important alternative directions of the study of integrable models like coordinate Bethe Ansatz [13], thermodynamic Bethe Ansatz [122, 123, 115, 124] and many others (see [2, 84, 114, 59] and references therein).

Let us continue with an overview of the algebraic Bethe Ansatz (ABA) method. This point of view gives an effective approach to deal with Eqs. (0.0.1) and (0.0.2). As pointed out in Section 0.1, the problem of diagonalising the Hamiltonian (0.1.4) is replaced by the one of finding the eigensystem of the transfer matrix T. The central

2. This Hamiltonian can be obtained by direct calculations using the formula (0.1.3), where the T-matrix is defined as the trace of the monodromy matrix (1.1.6) with R taken in the form (4.4.27).

object here is the IK R-matrix, $R \in \mathrm{End}(V \otimes V)$. Integrability is ensured by the Yang–Baxter equation

$$(I \otimes PR(\lambda/\mu))\,(PR(\lambda) \otimes I)\,(I \otimes PR(\mu)) = (PR(\mu) \otimes I)\,(I \otimes PR(\lambda))\,(PR(\lambda/\mu) \otimes I)\,.$$

The element I acts on the representation space V as the unit matrix and the Yang–Baxter (YB) equation takes place in $V \otimes V \otimes V$. The matrix $P \in \mathrm{End}(V \otimes V)$ simply interchanges the two factors of the tensor product. Due to the YB equation we have the commutativity (0.1.2) of the family of transfer matrices $T(t)$. Then we can turn to the algebraic Bethe Ansatz. As mentioned before, the transfer matrix is given as a trace of the monodromy matrix, while the commutativity of the monodromy matrix elements is mediated by the R-matrix. This gives rise to the Yang–Baxter algebra as the algebra of the matrix elements of the monodromy matrix. Furthermore, using this algebra one proposes a module on which the T-matrix has diagonal action. Since all participants are essentially in the YB algebra, one shows the validity of the eigenvalue equation with the T-matrix with the aid of the commutation relations of the YB algebra. The cost of this is a set of algebraic equations for a number of unknown complex numbers. These equations are known as the Bethe equations and the solution is called the set of Bethe roots. Thus, the problem of solving (0.0.1) is reduced to finding the Bethe roots. Let us postpone the discussion of the Bethe equations and pass to Eq. (0.0.2).

Physically interesting characteristics of quantum systems are obtained through the correlation functions. Therefore, one must have an efficient procedure for computing them. The algebraic Bethe Ansatz, i.e. the use of the YB algebra, is again a powerful tool here. How to use ABA in the study of correlation functions is described in [84, 111, 112] and in [78]. The latter case concerns the XXZ spin chain (the corresponding ABA was constructed in [41]). It focuses on finding a computationally convenient representation for the form factors (building blocks of the correlation functions). In the limit of infinite system size this representation leads to integral formulae for various correlation functions [66, 79, 77, 76, 75].

In general, however, one still needs to deal with the Bethe equations. Hence, the ABA method has to be supplemented by additional numerical or analytical tools. The numerical calculation of the Bethe roots and a clever algorithm of summing over the intermediate states can lead to a calculation of important physical quantities which can be measured in the experiment. For the integrable models such as XXZ spin chain, the δ-Bose gas and the Babujan–Takhtajan spin-1 chain we refer to [22, 23, 100, 110, 109, 104, 80, 40, 120]. A powerful analytical tool is provided by the theory of functional relations. The functional relations are the relations between the matrices which commute with the original transfer matrix. These matrices are also called transfer matrices. They are obtained as traces of monodromy matrices which have different representations along the so-called auxiliary space. By taking different representations π for the auxiliary space we arrive at different traces leading to a transfer matrix T_π. As it can be seen for example from the representation theory of the algebra associated to the model under consideration (see e.g. [86] and references therein), the transfer matrices for certain infinite class of representations are not independent; they satisfy bilinear algebraic equations, called the TT equations. Since the different transfer matrices commute with each other, the TT equations can be viewed as equations for the eigenvalues. The TT equations lead to the solution of the Bethe equations

[86] and allow one to calculate some physical quantities [87]. There exists a special class of transfer matrices, i.e. those in which the auxiliary space has certain infinite dimensional q-deformed bosonic representations [5, 8, 81, 14, 16, 15, 17, 98]. These transfer matrices are called the Q-matrices (or Q-operators). Among the Q-operators there are operators whose eigenvalues turn out to be the generating functions of the Bethe roots. The existence of such Q-operators was pointed out first by Baxter (see [3]). The diagonalization of the Q-operators essentially solves the Bethe equations. An example where an analytical calculation was performed towards the computation of a simple expectation value with the knowledge of the ground state eigenvalue of the Q-operator can be found in [56].

The above discussion is rather general. What is crucial in the algebraic Bethe Ansatz method is the assumption that the eigenvectors of the transfer matrix can be obtained by writing a module of the YB algebra. This is an advantage of the algebraic Bethe Ansatz: its universality. The functional relations are largely based on the representation theory and are often studied for many quantum affine Lie algebras at once [86, 88, 60, 61]. A broad class of integrable models can be studied in this way. One such model which is particularly well studied is the Heisenberg XXZ spin chain. This will be the reference model for us. For the XXZ spin chain both problems, the diagonalization of the transfer matrices and the calculation of the correlation functions, are well handled by means of the ABA.

0.3 Motivations

After the above discussion it follows that one of the first natural motivations for our study of the IK model is to try to find some of its physical characteristics, achieved by employing the described methods which, in particular, proved to be efficient for the XXZ model. The IK model appears to have a very complicated structure of the algebraic Bethe Ansatz. This brings many technical obstacles when trying to use the commutation relations of the YB algebra. Understanding how to overcome these obstacles is important if we want to "go beyond" the XXZ spin.

More generally, we are motivated to study the IK model since it describes a very rich physical system. It is related to a number of statistical models and, as we mentioned in the introduction, it is equivalent to the two dimensional IK nineteen vertex model on a square lattice. The IK nineteen vertex model is a classical statistical mechanics model. It would be interesting to study the thermodynamical properties of this model. In order to do that one could follow the example of the the six vertex model (the classical statistical model equivalent to the XXZ spin 1/2 chain) where the thermodynamics was studied using the integrability techniques. The six vertex model was solved in [93], see also [92, 94, 95] and the books [2, 114]. In particular, it was observed that the bulk free energy of the six vertex model is affected by the boundary conditions [82, 126]. The most famous case of such boundary conditions is the domain wall boundary. The dependence of the free energy on the boundary must also occur in the nineteen vertex model. We would also like to mention here the arctic curve phenomena appearing in the six vertex model in large square domains with the domain wall boundaries. The arctic curve separates the frozen and the disordered regions in the six vertex configurations (see papers [28, 29, 31, 30]). The arctic curve in its appropriately generalized sense must be also present in the nineteen vertex models. Integrability is one of the main tools for the study of the arctic curves as demonstrated in the six vertex example.

The configurations of the six vertex model with the domain wall boundary conditions are in one to one correspondence with an important combinatorial objects–the alternating sign matrices (ASM's) [89]. The discovery of this lead to solutions of the enumeration problems of ASM's [89, 90]. A generalization of such correspondence for the nineteen vertex models relates the configurations of the nineteen vertex model with the spin-1 ASM's [10]. Thus, the relation to the combinatorics of the higher spin ASM's is another motivation to study the IK model.

Let us get back to the connection of the IK model to statistical physics. One of the first found and most important connections is with the dilute Temperley–Lieb loop model [103, 102]. This loop model describes various two dimensional physical systems which, in particular, exhibit separation of phases by domain walls: percolation [48], polymer chains [102]. These two examples correspond to certain regimes with special interactions in the IK model. In these cases some quantities become more tractable and allow for explicit analytical calculations [48, 54, 57, 58, 42].

The ground state entries of the dense Temperley–Lieb loop model or the XXZ spin chain at a certain interaction point were found to be in relation to the combinatorics of the ASM's, as mentioned before. This led to various striking conjectures [1, 105, 107] that were later proved [49] and [19]. The main conjecture is called the Razumov–Stroganov (RS) correspondence, which became a theorem after [19]. The RS-type correspondence has not been observed yet in the dilute Temperley–Lieb loop model since it remains unclear what combinatorial object is related to the ground state entries of the dilute Temperley–Lieb model or the IK model.

An interesting relation appears between certain integrable models (IM) and ordinary differential equations (ODE) [38, 36, 6, 7, 97]. This relation is called the ODE/IM correspondence. For example, the Q-operator of the six vertex model with a twist κ corresponds to a solution $y(x)$ of the Schrödinger equation with the potential term $x^{2M(\kappa)}$. More generally the ODE/IM correspondence identifies the functional relations of integrable models with the Stokes relations of ordinary differential equations. It would be interesting to understand how ODE/IM works in the case of the Izergin-Korepin model. The first step in this direction was made in [37].

Finally, there also exists an important representation-theoretic aspect of the study of this model. The underlying algebra is the lowest rank twisted quantum affine Lie algebra, denoted $U_q(A_2^{(2)})$. Certain representations of this algebra are of physical interest which provides an interplay between the study of the representation theory of $U_q(A_2^{(2)})$ and the study of the IK model. One prominent example is the category of the prefundamental representations of the Borel subalgebras of quantum affine Lie algebras [62]. This study was motivated by the existence of the physically interesting infinite dimensional representations of certain quantum affine Lie algebras [5, 8, 81, 14, 16, 15, 17, 98]. A large part of this thesis is devoted to this aspect of the representation theory in the relation to the problem of the computation of the Q-operators.

0.4 Overview and outline

The R-matrix of the Izergin–Korepin nineteen vertex model is the central object. Using this R-matrix one constructs the monodromy matrix and writes the commutation relations of the generated Yang–Baxter algebra. There exists an algebraic Bethe Ansatz for the IK model due to Tarasov [116]. It provides a construction of the eigenvectors of the transfer matrix along with its eigenvalues and the associated Bethe Ansatz equations. The form of the resulting eigenvectors is very complicated, though. They are given by a recursive formula, which makes the problem of the computation of the correlation functions in the framework of the algebraic Bethe Ansatz very difficult. We present the algebraic Bethe Ansatz of Tarasov in Chapter 1. We propose a new way of writing the eigenvectors of the transfer matrix by solving the Tarasov's recurrence relation. It remains, however, an open question how to cast the ABA for the IK model in a form that allows for the efficient calculations of the form factors or correlation functions.

Following our reference model, the six vertex model, we address the question of the definition and computation of the domain wall partition function. This partition function plays a crucial role, in particular, in the theory of correlation functions for the six vertex model [84, 78]. An attempt to identify such object in the IK model is presented in Chapter 2. As a starting point we choose to consider the partition function of the nineteen vertex model with the domain wall boundary conditions. This partition function for a generic value of interaction is hard to write in a closed form. For a special interacting point we find a determinantal formula.

The representation theory, as we mentioned, is very important for the integrable models. In particular, one can find various transfer matrices associated to different representations along the auxiliary space of the monodromy matrix. In Chapter 3 we compute the higher dimensional Kirillov–Reshetikhin modules. In Chapter 4 this allows us to write the R matrices for the fundamental (three dimensional) and higher dimensional representations using the explicit form of the universal $\mathcal{R}$-matrix given by the Khoroshkin–Tolstoy formula [73, 117]. The original goal of Chapter 3 and Chapter 4 was to address the problem of solving the Bethe equations by means of the Baxter's Q-operator. According to the theory of prefundamental representations of Borel subalgebras [62] for the untwisted A-series quantum affine Lie algebras one must consider the infinite dimensional limit of the Kirillov–Reshetikhin modules. Taking this limit in the case of the $U_q(A_2^{(2)})$ algebra allows us to obtain certain infinite dimensional representations which then can be used in the Khoroshkin–Tolstoy formula to compute the L-matrices (R-matrices in a tensor product of the fundamental and an infinite dimensional representations). One obtains the Q-operators [3] by taking a trace of a product of the L-matrices. The Baxter's Q-operator should follow from this approach, unfortunately, we are currently unable to show this. We should note here, that as we learned recently a similar study was performed in a different spirit by M. Jimbo and J.-J. Sun. Our work is done independently from the work of Jimbo and Sun.

3. What we call here a Q-operator is different from the notion we used before. The Q-operator whose eigenvalues are the generating functions of the Bethe roots we call the Baxter's Q-operator. Otherwise a Q-operator is a transfer matrix obtained from a monodromy matrix with some infinite dimensional representation along its auxiliary space.

Chapter 5 is devoted to the study of the (conjectured) ground state at a root of unity. In this case we can solve the Bethe equations and find the entries of the ground state for small system sizes. These results agree with the ground state components obtained in the loop basis in [48, 54]. The connection between the spin and loop bases is described in Appendix C.

Chapter 1

Algebraic Bethe Ansatz for the IK model

In this chapter we discuss the algebraic Bethe Ansatz approach to the Izergin–Korepin model. This is a very convenient approach allowing for a construction of the eigenstates of the transfer matrix. The eigenstates live in the module generated by polynomials in the elements of the Yang–Baxter algebra which act on the reference state. The eigenstates and the eigenvalues constructed in the algebraic Bethe Ansatz depend on the set of Bethe roots. Therefore, the algebraic Bethe Ansatz reduces the problem of diagonalizing the transfer matrix to solving a system of algebraic equations whose solutions are the Bethe roots.

Our presentation of the algebraic Bethe Ansatz follows the paper of Tarasov [116]. We give it in Section 1.1 and 1.2. The eigenstates of the Tarasov's algebraic Bethe Ansatz are written using a recursive formula. In Section 1.3 we propose a new way of writing the eigenstates of the transfer matrix, which solves the Tarasov's recurrence relation. We write a formula which resembles the nested Bethe Ansatz for higher rank models. It is important to have a convenient formula for the eigenstates in order to study correlation functions in the framework of the algebraic Bethe Ansatz [84]. We discuss the scalar products in Section 1.4.

1.1 Introduction

The algebraic Bethe Ansatz (ABA) is a very powerful tool in the study of integrable models. In the ABA, instead of diagonalizing the Hamiltonian one attempts to diagonalize the family of matrices $T(\lambda)$ whose members are labeled by the spectral parameter λ. In the case of the IK spin chain one recasts the problem into a problem of diagonalizing the transfer matrix of the nineteen vertex model possessing the symmetry of the quantum group $U = U_q(A_2^{(2)}) = U_q(\hat{sl}_3^{\tau})$ [1]. This model is called the Izergin–Korepin (IK) nineteen vertex model [67] and we will refer to it as the IK model.

1. The notation $U_q(\hat{sl}_3^{\tau})$ means that the corresponding algebra is obtained by twisting the algebra $\hat{sl}_3$ with its Dynkin diagram automorphism τ. Thus the algebra $U_q(\hat{sl}_3^{\tau})$ belongs to the class of twisted quantum affine Lie algebras

The central object in the IK model is the R-matrix which depends on the spectral parameter λ, thus written $R(\lambda)$. There exist a universal $\mathcal{R}$-matrix which is an element of the algebra $U \otimes U$. If V is a fundamental representation of U, then the representation of $\mathcal{R}$ in $V \otimes V$ gives the R-matrix. We define the $\check{R}$-matrix as $\check{R}(\lambda) = PR(\lambda)$, where P permutes the spaces in the tensor product $V \otimes V$. The R-matrix is the solution of the Yang–Baxter equation, which is conveniently written in the $\check{R}$-form as [2]

$$\check{R}_{2,3}(\lambda-\mu)\check{R}_{1,2}(\lambda)\check{R}_{2,3}(\mu) = \check{R}_{1,2}(\mu)\check{R}_{2,3}(\lambda)\check{R}_{1,2}(\lambda-\mu), \tag{1.1.1}$$

where $\check{R}_{1,2}(x) = \check{R}(x) \otimes I$ and $\check{R}_{2,3}(x) = I \otimes \check{R}(x)$. The element I acts on the representation space V as the unit matrix and Eq. (1.1.1) takes place in $V \otimes V \otimes V$ and is a product of three matrices on both sides of the equality.

A quantum integrable system is characterized by the monodromy matrix $M(\lambda)$ which satisfies the following relation with the R-matrix

$$\check{R}.(\lambda-\mu)M(\lambda) \otimes M(\mu) = M(\mu) \otimes M(\lambda).\check{R}(\lambda-\mu). \tag{1.1.2}$$

The monodromy matrix M is a matrix in the space V, which is called the auxiliary space. If the monodromy matrix has a highest vector (pseudo vacuum or generating state) then the ABA is applicable. In the IK model, Tarasov [116] showed that such a pseudo vacuum exists and is represented by the ferromagnetic state $|0\rangle$. The ABA then provides us with the Yang–Baxter algebra which is the algebra generated by the matrix elements of the monodromy matrix with the relations encoded in (1.1.2). The representation space of the Yang–Baxter algebra is called the physical or quantum space. Taking the trace of the M-matrix in the auxiliary space yields the transfer matrix $T(\lambda)$ which is an element of the Yang–Baxter algebra. This is the transfer matrix that we wish to diagonalize. The pseudo vacuum is a special vector in the physical space which allows us to generate a module using certain elements of the Yang–Baxter algebra which play the role of creation and annihilation operators. This module contains the eigenvectors of the transfer matrix. The latter can be shown algebraically using the commutation relations of the Yang–Baxter algebra since both, the transfer matrix and the module itself, are formed by the Yang–Baxter algebra.

The Izergin–Korepin R-matrix, which satisfies (1.1.1) via $\check{R}$, is an operator in the tensor product of two fundamental representations of U. In the matrix form it reads

$$R(u) = \left[\begin{array}{ccc|ccc|ccc}
x_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & x_2 & 0 & x_5 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & x_3 & 0 & x_6 & 0 & x_7 & 0 & 0 \\
\hline
0 & y_5 & 0 & x_2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & y_6 & 0 & x_4 & 0 & x_6 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & x_2 & 0 & x_5 & 0 \\
\hline
0 & 0 & y_7 & 0 & y_6 & 0 & x_3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & y_5 & 0 & x_2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_1
\end{array}\right], \tag{1.1.3}$$

2. Eq. (1.1.1) is written in the additive convention while in the introduction we wrote the Yang–Baxter equation in the multiplicative convention. Later we will explain how to get from one convention to the other.

where the Boltzmann weights $x_j = x_j(u)$ and $y_j = y_j(u)$ are the following[3]

$$
\begin{aligned}
x_1(u) &= 2\sinh\left(\frac{u}{2} - 2\eta\right)\cosh\left(\frac{u}{2} - 3\eta\right), \\
x_2(u) &= 2\sinh\frac{u}{2}\cosh\left(\frac{u}{2} - 3\eta\right), \\
x_3(u) &= 2\sinh\frac{u}{2}\cosh\left(\frac{u}{2} - \eta\right), \\
x_4(u) &= 2\sinh\frac{u}{2}\cosh\left(\frac{u}{2} - 3\eta\right) - 2\sinh 2\eta\cosh 3\eta, \\
x_5(u) &= 2\mathrm{e}^{-\frac{1}{2}u}\sinh 2\eta\cosh\left(\frac{u}{2} - 3\eta\right), \\
y_5(u) &= -\mathrm{e}^{u}x_5(u), \\
x_6(u) &= 2\mathrm{e}^{-\frac{1}{2}u+2\eta}\sinh 2\eta\sinh\frac{u}{2}, \\
y_6(u) &= \mathrm{e}^{u-4\eta}x_6(u), \\
x_7(u) &= -2\mathrm{e}^{-\frac{1}{2}u}\sinh 2\eta[\cosh\left(\frac{u}{2} - 3\eta\right) + \mathrm{e}^{\eta}\sinh\frac{u}{2}], \\
y_7(u) &= -2\mathrm{e}^{\frac{1}{2}u}\sinh 2\eta[\cosh\left(\frac{u}{2} - 3\eta\right) - \mathrm{e}^{-\eta}\sinh\frac{u}{2}].
\end{aligned}
\tag{1.1.4}
$$

The physical space that we are interested in is a chain of length L where each site is the fundamental representation of U. Such a physical space is a tensor product $\mathcal{H}_L = V_1 \otimes \cdots \otimes V_L$ with L entries V_j each of which is $\mathbb{C}^3$. The auxiliary space is also $\mathbb{C}^3$, thus the monodromy matrix is a 3×3 matrix with entries being themselves operators in the physical space $\mathcal{H}_L$

$$
M(u) = \begin{bmatrix} A_1(u) & B_1(u) & B_2(u) \\ C_1(u) & A_2(u) & B_3(u) \\ C_2(u) & C_3(u) & A_3(u) \end{bmatrix}. \tag{1.1.5}
$$

The monodromy matrix can be viewed as follows. Let us equip the R-matrix with indices $R_{i,j}$ which indicate that it acts on the spaces V_i and V_j. In this notation $R_{i,i+1}$ acts as the R-matrix on the part $V_i \otimes V_{i+1}$ of the tensor product of the physical space $\mathcal{H}_L$ and on the remaining part of $\mathcal{H}_L$ it acts as the identity matrix. Let us denote the auxiliary space by V_0 which is also a copy of $\mathbb{C}^3$. The R-matrix $R_i(u) = R_{0,i}(u)$ acts on $V_0 \otimes \mathcal{H}_L$. We construct the following product

$$
M(u) = R_1(u)R_2(u)\ldots R_L(u), \tag{1.1.6}
$$

where the multiplication is assumed along the auxiliary space V_0. The matrix $M(u)$ is the monodromy matrix. This gives the construction of the monodromy matrices as products of the R-matrices along the neighbouring auxiliary spaces. It satisfies (1.1.2) due to the Yang–Baxter equation. The monodromy matrix M can be viewed as a matrix acting on V_0 and therefore it is a 3×3 matrix as in (1.1.5). Let us call the three

3. To obtain the original result [67] one needs to perform certain similarity transformation and change appropriately the parametrization, see e.g. [85].

basis vectors of V spin-up, spin-zero (empty edge) and spin-down, we will also use 1, 0 and -1, i.e. v_1, v_0 and v_{-1} respectively. The space V will be represented by a line and the R-matrix, which maps $V \otimes V$ to $V \otimes V$ will be represented by four lines (edges) which meet at a common point. Thus the elements of the R-matrix (1.1.3)

$$R(z_1/z_2) = \sum_{a,b,c,d=1,0,-1} r_{a,b}^{c,d}(z_1/z_2)\epsilon_{a,c} \otimes \epsilon_{b,d}, \tag{1.1.7}$$

where $\epsilon_{a,b}$ are the matrix units $\epsilon_{a,b}v_b = v_a$, can be represented graphically as in Fig. 1.1. This gives us the graphical notation which we will use in what follows. The arrows

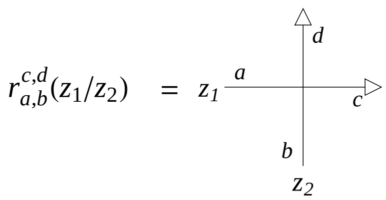


FIGURE 1.1 – The components $r_{a,b}^{c,d}(z_1/z_2)$.

in Fig. 1.1 distinguish the preimage $V \otimes V$ from the image $V \otimes V$ which has no arrows. Similarly the $\check{R}$ matrix has the expansion

$$\check{R}(z_1/z_2) = \sum_{a,b,c,d=1,0,-1} \check{r}_{a,b}^{c,d}(z_1/z_2)\epsilon_{a,c} \otimes \epsilon_{b,d}, \tag{1.1.8}$$

where the components $\check{r}_{a,b}^{c,d}(z_1/z_2)$ graphically are represented in Fig. 1.2. The relation

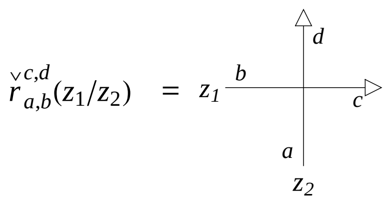


FIGURE 1.2 – The components $\check{r}_{a,b}^{c,d}(z_1/z_2)$.

between r and $\check{r}$ is

$$r_{a,b}^{c,d}(z_1/z_2) = \check{r}_{b,a}^{c,d}(z_1/z_2).$$

Each nonzero component of the R-matrix is represented by a vertex with edges equipped with arrows pointing towards the vertex, outwards the vertex or left free to denote the zero spin. The total number of the ingoing arrows must match the total number of the outgoing arrows. An edge corresponding to the preimage of the R-matrix has a value equal to 1, 0, or -1 if the arrow is outwards, absent or inwards, respectively. The reverse order of the arrows applies to the edges representing the image. In this

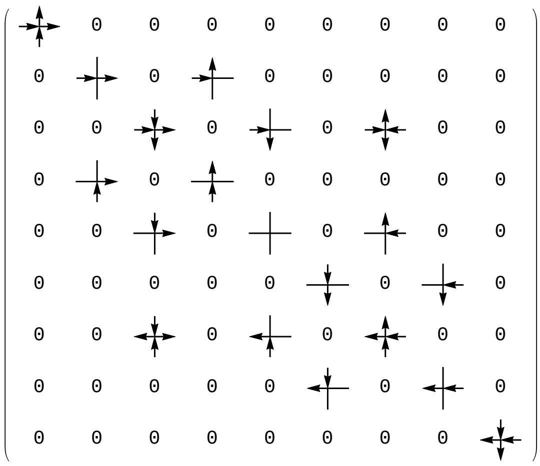


FIGURE 1.3 – The nineteen vertices of the R-matrix are the nonzero elements here. Their Boltzmann weights can be read off from Eq. (1.3).

language the components of the IK R-matrix can be represented by the nineteen vertices. The weights of the nineteen vertices are matched with the corresponding vertices by comparing Eq. (1.1.3) with Fig. 1.3.

Now we can also introduce a graphical notation for the monodromy matrix elements. These elements act on the physical space $\mathcal{H}_L$ thus must have $2L$ edges, L edges for the preimage and L for the image. In the auxiliary space V_0 the Yang–Baxter algebra elements correspond to the specific values of spin. The graphical representation of each element in (1.1.5) is shown in Fig. 1.4, Fig. 1.5 and Fig. 1.6.

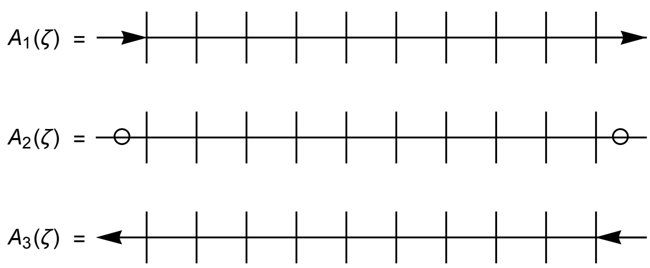


FIGURE 1.4 – Graphical representation of the A_1 (top), A_2 (middle) and A_3 (bottom) operators.

The transfer matrix is obtained from the monodromy matrix upon specifying the boundary conditions. Here we work with periodic boundary conditions with a twist which takes into account a vertical magnetic field. It is denoted by κ and for now it can be viewed as a free parameter. Tracing over the auxiliary space we obtain the transfer matrix

$$T_\kappa(u) = \kappa A_1(u) + A_2(u) + \kappa^{-1} A_3(u). \tag{1.1.9}$$

The twist κ will be important later, however, since it is normally easy to recover, we

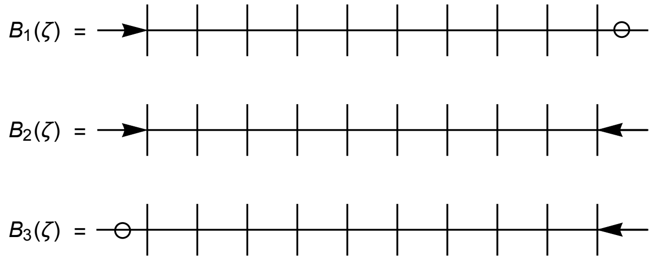


FIGURE 1.5 – Graphical representation of the B_1 (top), B_2 (middle) and B_3 (bottom) operators.

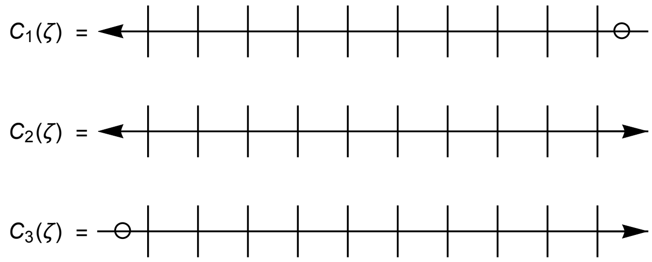


FIGURE 1.6 – Graphical representation of the C_1 (top), C_2 (middle) and C_3 (bottom) operators.

will omit it in this chapter and consider simply

$$T(u) = A_1(u) + A_2(u) + A_3(u). \tag{1.1.10}$$

Let us discuss now the two crucial components of the ABA: the Yang–Baxter algebra and the corresponding module built from the highest vector $|0\rangle$ of the monodromy matrix. The generators of the Yang–Baxter algebra are the entries of the matrix (1.1.5)

$$A_1(u),\ A_2(u),\ A_3(u),\ B_1(u),\ B_2(u),\ B_3(u),\ C_1(u),\ C_2(u),\ C_3(u). \tag{1.1.11}$$

The commutation relations of the Yang–Baxter algebra can be extracted from the relation (1.1.2) and using the explicit form of the R-matrix (1.1.3). Since there are plenty of such relations we prefer first to identify which ones are important for us and then write only those. First of all we define the normal ordering. The operators A_i must stand between the operators B_j and C_k, where B's are on the left side to the A's. It is not very important for us whether B_1 is on the right or on the left to B_2 or B_3 and similarly for A and C, what turns out to be important is the following. Each operator A_i, B_i or C_i depends on a spectral parameter ζ_j. Since any two operators with different spectral parameters do not commute in general, we choose to order them according to the label j of the parameter ζ_j. Two operators corresponding to the same letter, say B, ordered as $B_i(\zeta_s)B_j(\zeta_t)$ for $s < t$ and any i and j are normally ordered.

As we will see, the eigenvectors of the transfer matrix will be built by the action of a series of operators B_1 and B_2 on the pseudo vacuum. The transfer matrix is a sum of A-operators thus we need to write the commutation relations which will normal order

A's and B's and also any non ordered combination arising on the way. First we list the A's acting on B's

$$A_1(u)B_1(v) = \frac{x_1(v-u)B_1(v)A_1(u)}{x_2(v-u)} - \frac{x_5(v-u)B_1(u)A_1(v)}{x_2(v-u)}, \tag{1.1.12}$$

$$\begin{aligned} A_2(u)B_1(v) &= \frac{y_6(u-v)B_3(u)A_1(v)}{x_3(u-v)} + \frac{y_5(u-v)y_6(u-v)B_2(u)C_1(v)}{x_2(u-v)x_3(u-v)} \\ &+ B_1(v)A_2(u)\left(\frac{x_4(u-v)}{x_2(u-v)} - \frac{x_6(u-v)y_6(u-v)}{x_2(u-v)x_3(u-v)}\right) - B_1(u)A_2(v)\frac{y_5(u-v)}{x_2(u-v)} \\ &+ B_2(v)C_1(u)\left(\frac{x_6(u-v)}{x_2(u-v)} - \frac{x_7(u-v)y_6(u-v)}{x_2(u-v)x_3(u-v)}\right), \end{aligned} \tag{1.1.13}$$

$$\begin{aligned} A_3(u)B_1(v) &= -\frac{y_6(u-v)B_3(u)A_2(v)}{x_3(u-v)} + \frac{x_2(u-v)B_1(v)A_3(u)}{x_3(u-v)} \\ &+ \frac{x_5(u-v)B_2(v)C_3(u)}{x_3(u-v)} - \frac{y_7(u-v)B_2(u)C_3(v)}{x_3(u-v)}, \end{aligned} \tag{1.1.14}$$

$$\begin{aligned} A_1(u), B_2(v)x_3(v-u) &= -x_7(v-u)B_2(u)A_1(v) + x_1(v-u)B_2(v)A_1(u) \\ &- x_6(v-u)B_1(u)B_1(v), \end{aligned} \tag{1.1.15}$$

$$A_2(u)B_2(v)\frac{x_2(v-u)}{x_5(v-u)} = B_2(v)A_2(u)\frac{x_2(v-u)}{x_5(v-u)} - B_3(u)B_1(v) + B_3(v)B_1(u), \tag{1.1.16}$$

$$\begin{aligned} A_3(u)B_2(v)x_6(u-v) &= B_2(u)A_3(v)\left(-\frac{x_1(u-v)\omega(v-u)x_6(v-u)}{x_3(v-u)} - y_6(u-v)\right) \\ &+ B_2(v)A_3(u)\left(\frac{x_1(u-v)x_6(v-u)y_7(v-u)\omega(v-u)}{x_1(v-u)x_3(v-u)} - \frac{x_1(u-v)y_6(v-u)\omega(v-u)}{x_1(v-u)}\right) \\ &+ B_3(u)B_3(v)\left(x_1(u-v)\omega(v-u) - x_4(u-v)\right), \end{aligned} \tag{1.1.17}$$

$$\begin{aligned} A_1(u)B_3(v) &= \frac{x_2(v-u)B_3(v)A_1(u)}{x_3(v-u)} - \frac{x_6(v-u)B_1(u)A_2(v)}{x_3(v-u)} \\ &- \frac{x_7(v-u)B_2(u)C_1(v)}{x_3(v-u)} + \frac{y_5(v-u)B_2(v)C_1(u)}{x_3(v-u)}, \end{aligned} \tag{1.1.18}$$

$$\begin{aligned} A_2(u)B_3(v) &= B_3(v)A_2(u)\left(\frac{x_4(v-u)}{x_2(v-u)} - \frac{x_6(v-u)y_6(v-u)}{x_2(v-u)x_3(v-u)}\right) \\ &- \frac{x_5(v-u)B_3(u)A_2(v)}{x_2(v-u)} + \frac{x_6(v-u)B_1(u)A_3(v)}{x_3(v-u)} \\ &+ \frac{x_5(v-u)x_6(v-u)B_2(u)C_3(v)}{x_2(v-u)x_3(v-u)} \\ &+ B_2(v)C_3(u)\left(\frac{y_6(v-u)}{x_2(v-u)} - \frac{x_6(v-u)y_7(v-u)}{x_2(v-u)x_3(v-u)}\right), \end{aligned} \tag{1.1.19}$$

$$A_3(u)B_3(v) = \frac{x_1(u-v)B_3(v)A_3(u)}{x_2(u-v)} - \frac{y_5(u-v)B_3(u)A_3(v)}{x_2(u-v)}, \tag{1.1.20}$$

where we introduced

$$\omega(v) = \frac{x_1(v)x_3(v)}{x_3(v)x_4(v) - x_6(v)y_6(v)}. \tag{1.1.21}$$

The operators C_1 and C_3 appeared above, so we need to write some of the commutation relations between C and B

$$C_1(u)B_1(v)\frac{x_2(v-u)}{x_5(v-u)} = -A_2(u)A_1(v) + A_2(v)A_1(u) + B_1(v)C_1(u)\frac{x_2(v-u)}{x_5(v-u)}. \tag{1.1.22}$$

$$\begin{aligned}
C_1(u)B_2(v) &= \frac{x_6(v-u)y_5(u-v)B_1(u)A_2(v)}{x_2(u-v)x_3(v-u)} - \frac{x_2(v-u)y_5(u-v)B_3(v)A_1(u)}{x_2(u-v)x_3(v-u)} \\
&+ \frac{(x_3(u-v)x_3(v-u)y_6(u-v) - x_3(v-u)x_6(u-v)y_7(u-v))\,B_1(v)A_2(u)}{x_2(u-v)x_3(u-v)x_3(v-u)} \\
&+ \frac{(x_3(u-v)x_7(v-u)y_5(u-v) + x_3(v-u)y_7(u-v)y_5(u-v))\,B_2(u)C_1(v)}{x_2(u-v)x_3(u-v)x_3(v-u)} \\
&+ \frac{(-x_3(u-v)y_5(u-v)y_5(v-u) - x_3(v-u)x_7(u-v)y_7(u-v) + x_3(v-u)x_3^2(u-v))}{x_2(u-v)x_3(u-v)x_3(v-u)} \\
&\times B_2(v)C_1(u) + +\frac{y_7(u-v)B_3(u)A_1(v)}{x_3(u-v)},
\end{aligned} \tag{1.1.23}$$

$$\begin{aligned}
x_2(v-u)C_2(u)B_1(v) &= -x_5(v-u)C_3(u)A_1(v) + x_6(v-u)A_2(v)C_1(u) \\
&+ x_7(v-u)C_3(v)A_1(u) + x_3(v-u)B_1(v)C_2(u)
\end{aligned} \tag{1.1.24}$$

$$\begin{aligned}
x_3(u-v)C_3(u)B_1(v) &= y_6(u-v)(A_1(v)A_3(u) - A_2(u)A_2(v)) \\
&+ x_4(u-v)B_1(v)C_3(u) + x_6(u-v)B_2(v)C_2(u) - y_7(u-v)B_1(u)C_3(v).
\end{aligned} \tag{1.1.25}$$

$$\begin{aligned}
C_3(u)B_2(v) &= \frac{(x_3(u-v)x_3(v-u)x_6(v-u) - x_3(u-v)x_7(v-u)y_6(v-u))\,B_3(v)A_2(u)}{x_2(v-u)x_3(u-v)x_3(v-u)} \\
&+ \frac{x_5(v-u)y_6(u-v)B_3(u)A_2(v)}{x_2(v-u)x_3(u-v)} - \frac{x_2(u-v)x_5(v-u)B_1(v)A_3(u)}{x_2(v-u)x_3(u-v)} \\
&+ \frac{(x_3(v-u)x_5(v-u)y_7(u-v) + x_3(u-v)x_5(v-u)x_7(v-u))}{x_2(v-u)x_3(u-v)x_3(v-u)}B_2(u)C_3(v) \\
&- \frac{(x_3(u-v)x_7(v-u)y_7(v-u) - x_3(u-v)x_3^2(v-u) + x_5(u-v)x_5(v-u)x_3(v-u))}{x_2(v-u)x_3(u-v)x_3(v-u)} \\
&\times B_2(v)C_3(u) + \frac{x_7(v-u)B_1(u)A_3(v)}{x_3(v-u)}.
\end{aligned} \tag{1.1.26}$$

We need also some commutation relations between the B's.

$$\begin{aligned} B_1(u)B_1(v) = \omega(v-u)\left(B_1(v)B_1(u) - \frac{1}{y(v-u)}B_2(v)A_1(u)\right) \\ + \frac{1}{y(u-v)}B_2(u)A_1(v), \end{aligned} \tag{1.1.27}$$

$$x_1(v-u)B_2(v)B_1(u) = x_2(v-u)B_1(u)B_2(v) + x_5(v-u)B_2(u)B_1(v), \tag{1.1.28}$$

$$x_2(v-u)[B_1(u),B_3(v)] = -x_5(v-u)B_2(u)A_2(v) + y_5(v-u)B_2(v)A_2(u), \tag{1.1.29}$$

$$\begin{aligned} x_1(v-u)x_2(u-v)B_2(u)B_3(v) = -x_2(v-u)x_5(u-v)B_2(v)B_3(u) \\ + \left(x_1(u-v)x_1(v-u) - x_5(u-v)x_5(v-u)\right)B_3(v)B_2(u). \end{aligned} \tag{1.1.30}$$

There are many more relations, however, we must stop here since this will be enough for our purposes.

1.2 Tarasov's construction

In the paper [116] Tarasov showed how to construct the eigenvectors of the transfer matrix (also see [96]). The highest vector of the monodromy matrix has the following properties

$$C_i(v)|0\rangle = 0, \quad A_i(v)|0\rangle = \alpha_i(v)|0\rangle, \quad B_i(v)|0\rangle \neq 0. \tag{1.2.1}$$

Where the explicit form of α_i in the homogeneous and inhomogeneous models reads respectively

$$\alpha_i(\zeta) = x_i^L(\zeta), \quad \alpha_i(\zeta) = \prod_{j=1}^{L} x_i(\zeta - z_j). \tag{1.2.2}$$

The parameter ζ is associated to the horizontal line in Fig. 1.4 while the inhomogeneities z_j are associated to the vertical lines of the physical space $\mathcal{H}_L$. We will work in the homogeneous setting in this chapter while in later chapters we restore the inhomogeneities. Since the pseudo vacuum is an eigenstate of the operators A_1, A_2 and A_3, it is also an eigenstate of the transfer matrix. The pseudo vacuum is the fully ferromagnetic state (say, all spins-up) and, therefore, is also called the state with zero particles, or magnons. By acting on this state with an operator B_1 or B_3 we obtain a state which is a linear combination of states with an extra empty edge. This extra empty edge plays a role of a particle, thus such a state is called a one particle state. Acting with the operator B_2 we overturn one spin, creating two particles. On the other hand, two particles can be produced by acting twice with any of the two operators B_1, B_3. Thus two particles can be viewed either as two empty edges or an overturned spin (spin-down). The operators C_1 and C_3 act in the opposite way, their image must have a deficiency of an empty edge with respect to the preimage, while C_2 adds a spin-up to the preimage. It is clear now why these operators annihilate the fully ferromagnetic state (1.2.1). A state with N spins pointing downwards is called an N-particle state. We can group all states according to the number N into the N-particle sectors. Tarasov showed in [116] how to find a unique eigenvector in each sector. Clearly, to get a state with higher N we must act with the B operators. Equipping each operator of the monodromy matrix with the grading (called order in [116]) according to the number of particles they introduce to the system

$$\operatorname{grad}\begin{bmatrix} A_1(u) & B_1(u) & B_2(u) \\ C_1(u) & A_2(u) & B_3(u) \\ C_2(u) & C_3(u) & A_3(u) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & 1 \\ -2 & -1 & 0 \end{bmatrix}, \tag{1.2.3}$$

we can see that the states in the N-particle sector are created by the polynomials in the YB algebra elements whose monomials all have the total degree equal to N. The transfer matrix eigenstates are then written as

$$|\Psi_N(\zeta_1,..,\zeta_N)\rangle = \Phi_N(\zeta_1,..,\zeta_N)|0\rangle. \tag{1.2.4}$$

Where Φ_N is a certain polynomial in the YB algebra elements. The following symmetry condition for $\Phi_N(\zeta_1,..,\zeta_N)$ is very important

$$\Phi_N(\zeta_1,..,\zeta_{i+1},\zeta_i,..,\zeta_N) = \omega(\zeta_i - \zeta_{i+1})\Phi_N(\zeta_1,..,\zeta_i.\zeta_{i+1},..,\zeta_N). \tag{1.2.5}$$

This condition plays a crucial role in the derivation of the recurrence relation for Φ_N. We refer to [116, 96] for more details. The answer is written in the following form

$$\Phi_N(\zeta_1,..,\zeta_N) = B_1(\zeta_1)\Phi_{N-1}(\zeta_2,..,\zeta_N) + B_2(\zeta_1)\sum_{i>1} c_{1,i}(\zeta_1,..,\zeta_N)\Phi_{N-2}(\zeta_2,..,\hat{\zeta}_i,..,\zeta_N), \tag{1.2.6}$$

where ζ's satisfy the Bethe equations which we will write below and the variables with the hat $\hat{\zeta}_i$ are absent from the corresponding list. The initial conditions are $\Phi_0 = 1$ and $\Phi_1 = B_1(\zeta_1)$. The coefficients $c_{l,i}$ are

$$c_{l,i} = -\prod_{m\geq i}^{n} x_1(\zeta_i - z_m)\sum_{j>l}\frac{1}{y(\zeta_l - \zeta_j)}\prod_{k>l,k\neq j}^{n} Z(\zeta_k - \zeta_j), \tag{1.2.7}$$

where we explicitly wrote the dependence on the inhomogeneities z_i's and made use of the following shorthand notations

$$z(\zeta) = \frac{x_1(\zeta)}{x_2(\zeta)}, \quad y(\zeta) = \frac{x_3(\zeta)}{y_6(\zeta)}, \tag{1.2.8}$$

$$Z(\zeta_k - \zeta_j) = \begin{cases} z(\zeta_k - \zeta_j) & \text{if } k > j, \\ z(\zeta_k - \zeta_j)\omega(\zeta_j - \zeta_k) & \text{if } k < j. \end{cases} \tag{1.2.9}$$

Recall that the physical space $\mathcal{H}_L$ is composed of L copies of $\mathbb{C}^3$. The example of the graphical representation of the 3-particle eigenstate in $\mathcal{H}_4$ is shown in Fig. 1.7.

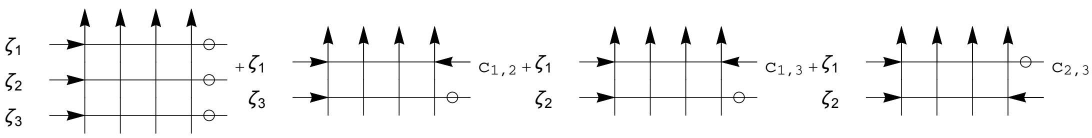


FIGURE 1.7 – The three particle state $|\Psi_4(\zeta_1,\zeta_2,\zeta_3)\rangle$ for the system size $L = 4$.

One can show using Eqs. (1.1.12)-(1.1.30) that the vector (1.2.4) with Φ_N defined in (1.2.6) is an eigenstate of the transfer matrix. Below we partially omit the dependence on the variables ζ_i and assume $\Phi_N = \Phi_N(\zeta_1,..,\zeta_N)$, moreover, the notation $\Phi_{N-1}[\zeta_i]$ will mean that ζ_i is absent from the list $(\zeta_1,..,\zeta_N)$ and equally $\Phi_{N-2}[\zeta_i,\zeta_j]$ means that ζ_i and ζ_j are absent from the same list. First, act with $A_1(\zeta)$ on (1.2.6). According to the chosen normal ordering we need to commute the A-operators to the right, using the YB algebra (1.1.12)-(1.1.30) we get

$$\begin{aligned}
A_1(\zeta)\Phi_N &= \prod_{k=1}^{N} z(\zeta_k-\zeta)\Phi_N A_1(\zeta)\\
&-B_1(\zeta)\sum_{j=1}^{N}\frac{x_5(\zeta_j-\zeta)}{x_2(\zeta_j-\zeta)}\prod_{k=1,k\neq j}^{N} Z(\zeta_k-\zeta_j)\Phi_{N-1}[\zeta_j]A_1(\zeta_j)\\
&+B_2(\zeta)\sum_{j=2}^{N}\sum_{l=1}^{j-1}G_{jl}(\zeta,\zeta_l,\zeta_j)\prod_{k=1,k\neq j,l}^{N} Z(\zeta_k-\zeta_l)Z(\zeta_k-\zeta_j)\\
&\times\Phi_{N-2}[\zeta_l,\zeta_j]A_1(\zeta_l)A_1(\zeta_j)\ ,
\end{aligned}\tag{1.2.10}$$

where $G_{jl}(\zeta,\zeta_l,\zeta_j)$ are defined by

$$G_{jl}(\zeta,\zeta_l,\zeta_j) = \frac{x_7(\zeta_l-\zeta)}{x_3(\zeta_l-\zeta)}\frac{1}{y(\zeta_l-\zeta_j)} + \frac{z(\zeta_l-\zeta)}{\omega(\zeta_l-\zeta)}\frac{x_5(\zeta_j-\zeta)}{x_2(\zeta_j-\zeta)}\frac{1}{y(\zeta-\zeta_l)}. \tag{1.2.11}$$

Similarly, the action of $A_2(\zeta)$ on Φ_N reads

$$\begin{aligned}
A_2(\zeta)\Phi_N &= \prod_{k=1}^{N}\frac{z(\zeta-\zeta_k)}{\omega(\zeta-\zeta_k)}\Phi_N A_2(\zeta)\\
&- B_1(\zeta)\sum_{j=1}^{N}\frac{y_5(\zeta-\zeta_j)}{x_2(\zeta-\zeta_j)}\prod_{k=1,k\neq j}^{N} Z(\zeta_j-\zeta_k)\Phi_{N-1}[\zeta_j]A_2(\zeta_j)\\
&+ B_3(\zeta)\sum_{j=1}^{N}\frac{1}{y(\zeta-\zeta_j)}\prod_{k=1,k\neq j}^{N} Z(\zeta_k-\zeta_j)\Phi_{n-1}[\zeta_j]A_1(\zeta_j)\\
&+ B_2(\zeta)\sum_{j=2}^{N}\sum_{l=1}^{j-1}Y_{jl}(\zeta,\zeta_l,\zeta_j)\prod_{k=1,k\neq j,l}^{N} Z(\zeta_k-\zeta_l)Z(\zeta_j-\zeta_k)\\
&\times\Phi_{N-2}[\zeta_l,\zeta_j]A_1(\zeta_l)A_2(\zeta_j)\\
&+ B_2(\zeta)\sum_{j=2}^{N}\sum_{l=1}^{j-1}F_{jl}(\zeta,\zeta_l,\zeta_j)\prod_{k=1,k\neq j,l}^{N} Z(\zeta_l-\zeta_k)Z(\zeta_k-\zeta_j)\\
&\times\Phi_{N-2}[\zeta_l,\zeta_j]A_1(\zeta_j)A_2(\zeta_l).
\end{aligned}\tag{1.2.12}$$

The functions $F_{jl} = F_{jl}(\zeta,\zeta_l,\zeta_j)$ and $Y_{jl} = Y_{jl}(\zeta,\zeta_l,\zeta_j)$ are the following

$$F_{jl} = \frac{y_5(\zeta-\zeta_l)}{x_2(\zeta-\zeta_l)}\frac{1}{y(\zeta-\zeta_l)}\Big(\frac{y_5(\zeta_l-\zeta_j)}{x_2(\zeta_l-\zeta_j)} + \frac{z(\zeta-\zeta_l)}{\omega(\zeta-\zeta_l)} - \frac{y_5(\zeta-\zeta_l)}{x_2(\zeta-\zeta_l)}\frac{y(\zeta-\zeta_l)}{y(\zeta_l-\zeta_j)}\Big), \quad (1.2.13)$$

$$Y_{jl} = \frac{1}{y(\zeta-\zeta_l)}\Big(z(\zeta-\zeta_l)\frac{y_5(\zeta-\zeta_j)}{x_2(\zeta-\zeta_j)} - \frac{y_5(\zeta-\zeta_l)}{x_2(\zeta-\zeta_l)}\frac{y_5(\zeta_l-\zeta_j)}{x_2(\zeta_l-\zeta_j)}\Big). \quad (1.2.14)$$

Finally, the action of $A_3(\zeta)$ gives

$$\begin{aligned} A_3(\zeta)\Phi_N &= \prod_{k=1}^{N}\frac{x_2(\zeta-\zeta_k)}{x_3(\zeta-\zeta_k)}\Phi_N A_3(\zeta) \\ &-B_3(\zeta)\sum_{j=1}^{N}\frac{1}{y(\zeta-\zeta_j)}\prod_{k=1,k\neq j}^{N} Z(\zeta_j-\zeta_k)\Phi_{N-1}[\zeta_j]A_2(\zeta_j) \\ &+B_2(\zeta)\sum_{j=2}^{N}\sum_{l=1}^{j-1}H_{jl}(\zeta,\zeta_l,\zeta_j)\prod_{k=1,k\neq j,l}^{N} Z(\zeta_j-\zeta_k)Z(\zeta_l-\zeta_k) \\ &\times \Phi_{N-2}[\zeta_l,\zeta_j]A_2(\zeta_l)A_2(\zeta_j), \end{aligned} \quad (1.2.15)$$

where $H_{jl}(\zeta,\zeta_l,\zeta_j)$ is given by

$$H_{jl}(\zeta,\zeta_l,\zeta_j) = \frac{y_7(\zeta-\zeta_l)}{x_3(\zeta-\zeta_l)}\frac{1}{y(\zeta_l-\zeta_j)} - \frac{y_5(\zeta-\zeta_l)}{x_3(\zeta-\zeta_l)}\frac{1}{y(\zeta-\zeta_j)}. \quad (1.2.16)$$

Summing up (1.2.10), (1.2.12) and (1.2.15) one finds that $\Phi_N(\zeta_1,\ldots,\zeta_N)|0\rangle$ is an eigenstate of the transfer matrix $T(\zeta)$ with the eigenvalue

$$\Lambda_N(\zeta) = x_1(\zeta)^L\prod_{a=1}^{N} z(\zeta_a-\zeta) + x_2(\zeta)^L\prod_{a=1}^{N}\frac{z(\zeta-\zeta_a)}{\omega(\zeta-\zeta_a)} + x_3(\zeta)^L\prod_{a=1}^{N}\frac{x_2(\zeta-\zeta_a)}{x_3(\zeta-\zeta_a)}, \quad (1.2.17)$$

which holds if the numbers ζ_i, $i = 1,\ldots,N$ satisfy the Bethe Ansatz equations (BAE)

$$\Big(\frac{x_1(\zeta_a)}{x_2(\zeta_a)}\Big)^L = \prod_{b\neq a=1}^{N}\frac{z(\zeta_a-\zeta_b)}{z(\zeta_b-\zeta_a)}\omega(\zeta_b-\zeta_a), \quad a = 1,2,\ldots,N. \quad (1.2.18)$$

In order to pass to the inhomogeneous model we simply replace the terms $x_i(\zeta)^L$ with the products as in (1.2.2).

1.3 Rewriting Tarasov's eigenstate

Recalling the nested Bethe Ansatz we may try to rewrite Eq. (1.2.6) in a product form. This means we need to embed our model in a larger space. It can be done as follows. Consider a N-particle state and define the new operators $\beta(\zeta_i|\zeta_{i+1},..\zeta_N)$ as

$$\beta(\zeta_i|\zeta_{i+1},..,\zeta_N) = \mathbb{I} + B_1(\zeta_i)f_i + B_2(\zeta_i)\times\sum_{j>i} c_{i,j}f_jf_i, \quad (i = 1,..N), \quad (1.3.1)$$

where $c_{i,j}$'s are the same as before (1.2.7), $\mathbb{I}$ and f_i are elements of some algebra with $\mathbb{I}$ being the identity and f_i ($i = 1,..,N$) obeying the following properties

$$[f_i,f_j] = 0, \quad f_i^2 = 0. \tag{1.3.2}$$

Taking the product of β_i's we get an eigenstate of the transfer matrix, which, as we will argue below, is a sum of the eigenstates of the sectors with $0,1,..,N$ particles. The f's give the grading to the terms in this sum. Choosing the terms of degree j in f we will get the eigenstate with j particles. We apply to this product the modified pseudovacuum $|0\rangle \otimes |\tilde{0}\rangle$, where $|\tilde{0}\rangle$ and its dual will serve to us as projectors to the polynomials of the degree (in the sense of (1.2.3)) that we ask. The state $|\tilde{0}\rangle = |\tilde{0}\rangle_N$ ant its dual ${}_N\langle\tilde{0}|$ are such that

$${}_N\langle\tilde{0}| \prod_{i=1}^{n} f_{a_i} |\tilde{0}\rangle_N = \delta_{n,N}. \tag{1.3.3}$$

The eigenstate $|\Psi_N\rangle$ can be written in terms of the operators β as

$$|\Psi_N(\zeta_1,..,\zeta_N)\rangle =_N \langle\tilde{0}| \prod_{i=1}^{N} \beta(\zeta_i|\zeta_{i+1},..,\zeta_N)|\tilde{0}\rangle_N \otimes |0\rangle. \tag{1.3.4}$$

We can write another expression for this eigenstate avoiding the operators f_i but including instead a contour integration and replacing the product of β's by an exponential as follows

$$|\Psi_N\rangle = \oint \frac{\mathrm{d}x}{x^{N+1}} : \exp\left(x^2 \sum_{1\le i<j\le N} c_{i,j} B_2(\zeta_i) + x \sum_{1\le i\le N} B_1(\zeta_i) \right) : |0\rangle, \tag{1.3.5}$$

or

$$|\Psi_N\rangle = \oint \frac{\mathrm{d}x}{x^{N+1}} : \prod_{i=1}^{N} e^{x\mathcal{B}(\zeta_i;x)} : |0\rangle,$$
$$\mathcal{B}(\zeta_i;x) = xB_2(\zeta_i) \sum_{i<j\le N} c_{i,j} + B_1(\zeta_i).$$

The contour goes around the point $x = 0$. The normal ordering " $:$ " here is as we defined above which coincides with that of [116] with the additional condition

$$X(\zeta_j)^2 = 0, \quad \text{for } X = A_i,\, B_i,\, C_i, \quad i = 1,2,3. \tag{1.3.6}$$

This condition must be understood as a rule of the normal ordering, of course, the actual matrices $A_i(\zeta_j)$, $B_i(\zeta_j)$ and $C_i(\zeta_j)$ do not square to zero. Let us mention a few realizations of the $\{\mathbb{I},f_i\}$ algebra. In the representation space $\mathbb{C}^{2^{\otimes N}}$, f_i is simply the Pauli σ^+ (or σ^-) matrix acting on the i-th space of the tensor product, and $\mathbb{I}$ is the identity matrix.

$$f_i = \sigma_i^+ = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes .. \otimes \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}_i \otimes .. \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{1.3.7}$$

The nontrivial part of the operators f_i in this representation is the Paul matrix σ^+ acting on the i-th tensor component of the space $\mathbb{C}^{2^{\otimes N}}$. Hence f_i and f_j commute when $i \neq j$ and since the Pauli matrices are nilpotent both conditions in (1.3.2) are satisfied.

The vacuum state $|\tilde{0}\rangle_N$ will become

$$|\tilde{0}\rangle_N = \begin{pmatrix} 1 \\ 0 \end{pmatrix}_1 \otimes .. \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}_N, \tag{1.3.8}$$

and in the dual state the zero's and one's must be interchanged. Another two realizations will be

$$f_i = \frac{\partial}{\partial w_i}, \quad |\tilde{0}\rangle_N = \prod_{i=1}^N w_i, \quad {}_N\langle\tilde{0}| = \oint \prod_{i=1}^N \frac{\mathrm{d}w_i}{w_i}, \quad \text{or} \tag{1.3.9}$$

$$f_i = \frac{1}{w_i}, \quad |\tilde{0}\rangle_N = 1, \quad {}_N\langle\tilde{0}| = \oint \prod_{i=1}^N \mathrm{d}w_i. \tag{1.3.10}$$

Where the contours go around $w_i = 0$. In both realisations we must also require the condition $f_i^2 = 0$. This condition need not be separately required if we consider the operators f_i acting on the vacuum in the first case (1.3.9) and on the dual vacuum in the second case (1.3.10)

$$\frac{\partial^2}{\partial w_j^2} \prod_{i=1}^N w_i = 0, \qquad \oint \prod_{i=1}^N \mathrm{d}w_i \frac{1}{w_j^2} = 0.$$

These two realisations are related to the exponential form (1.3.5). Let us now use, say (1.3.9), to show that the product of β in (1.3.4) satisfies the Tarasov's recurrence (1.2.6).

$${}_N\langle\tilde{0}| \prod_{i=1}^N \beta_i(\zeta_i) |\tilde{0}\rangle_N = \oint \prod_{i=1}^N \frac{\mathrm{d}w_i}{w_i} \prod_{i=1}^N \left(\mathbb{I} + B_1(\zeta_i) \frac{\partial}{\partial w_i} + B_2(\zeta_i) \sum_{j>i} c_{i,j} \frac{\partial}{\partial w_j} \frac{\partial}{\partial w_i} \right) \prod_{i=1}^N w_i,$$

where $\beta_i(\zeta_i) = \beta(\zeta_i|\zeta_{i+1},..,\zeta_N)$. Next we isolate the first multiplier $\beta_1(\zeta_1)$ in the product in the integrand, and expanding it we get three terms

$$\begin{aligned}
&\oint \prod_{i=1}^N \frac{\mathrm{d}w_i}{w_i} \mathbb{I} \prod_{i=2}^N \left(\mathbb{I} + B_1(\zeta_i) \frac{\partial}{\partial w_i} + B_2(\zeta_i) \sum_{j>i} c_{i,j} \frac{\partial}{\partial w_j} \frac{\partial}{\partial w_i} \right) \prod_{i=1}^N w_i \\
&+ \oint \prod_{i=1}^N \frac{\mathrm{d}w_i}{w_i} B_1(\zeta_1) \frac{\partial}{\partial w_1} \prod_{i=2}^N \left(\mathbb{I} + B_1(\zeta_i) \frac{\partial}{\partial w_i} + B_2(\zeta_i) \sum_{j>i} c_{i,j} \frac{\partial}{\partial w_j} \frac{\partial}{\partial w_i} \right) \prod_{i=1}^N w_i \\
&+ \oint \prod_{i=1}^N \frac{\mathrm{d}w_i}{w_i} B_2(\zeta_1) \sum_{j>1} c_{1,j} \frac{\partial}{\partial w_j} \frac{\partial}{\partial w_1} \prod_{i=2}^N \left(\mathbb{I} + B_1(\zeta_i) \frac{\partial}{\partial w_i} + B_2(\zeta_i) \sum_{j>i} c_{i,j} \frac{\partial}{\partial w_j} \frac{\partial}{\partial w_i} \right) \prod_{i=1}^N w_i.
\end{aligned}$$

In the first term the product which contains the B operators does not depend on w_1, hence w_1 in the denominator is cancelled with w_1 coming from $|\tilde{0}\rangle_N$ and the whole integrand does not depend on w_1. Therefore the first term is equal to zero. In a similar way we can also integrate the second and the third terms with respect to w_1. For the second term this integration gives

$$\begin{aligned}
&B_1(\zeta_1) \oint \prod_{i=2}^{N} \frac{\mathrm{d}w_i}{w_i} \prod_{i=2}^{N} \left(\mathbb{I} + B_1(\zeta_i) \frac{\partial}{\partial w_i} + B_2(\zeta_i) \sum_{j>i} c_{i,j} \frac{\partial}{\partial w_j} \frac{\partial}{\partial w_i} \right) \prod_{i=2}^{N} w_i \\
&= B_1(\zeta_1)_{N-1} \langle \tilde{0} | \prod_{i=2}^{N} \beta_i(\zeta_i) | \tilde{0} \rangle_{N-1},
\end{aligned} \tag{1.3.11}$$

and for the third term

$$\begin{aligned}
&B_2(\zeta_1) \sum_{j>1} c_{1,j} \oint \prod_{i=2}^{N} \frac{\mathrm{d}w_i}{w_i} \frac{\partial}{\partial w_j} \prod_{i=2}^{N} \left(\mathbb{I} + B_1(\zeta_i) \frac{\partial}{\partial w_i} + B_2(\zeta_i) \sum_{k>i} c_{i,k} \frac{\partial}{\partial w_k} \frac{\partial}{\partial w_i} \right) \prod_{i=2}^{N} w_i \\
&= B_2(\zeta_1) \sum_{j>1} c_{1,j} \oint \prod_{\substack{i=2 \\ i\neq j}}^{N} \frac{\mathrm{d}w_i}{w_i} \prod_{\substack{i=2 \\ i\neq j}}^{N} \left(\mathbb{I} + B_1(\zeta_i) \frac{\partial}{\partial w_i} + B_2(\zeta_i) \sum_{\substack{k>i, \\ k\neq j}} c_{i,k} \frac{\partial}{\partial w_k} \frac{\partial}{\partial w_i} \right) \prod_{\substack{i=2, \\ i\neq j}}^{N} w_i \\
&= B_2(\zeta_1) \sum_{j>1} c_{1,j} \ {}_{N-2}\langle \tilde{0} | \prod_{\substack{i=2 \\ i\neq j}}^{N} \beta_i^{(j)}(\zeta_i) | \tilde{0} \rangle_{N-2},
\end{aligned} \tag{1.3.12}$$

where the superscript (j) in $\beta_i^{(j)}(\zeta_i)$ means that $\beta_i^{(j)}(\zeta_i) = \beta(\zeta_i|,\zeta_{i+1},..,\hat{\zeta}_j,..,\zeta_N)$ has no dependence on ζ_j (for $i > j$ this holds automatically). Gathering (1.3.11) and (1.3.12) together we obtain

$$\begin{aligned}
{}_N\langle \tilde{0} | \prod_{i=1}^{N} \beta_i(\zeta_i) | \tilde{0} \rangle_N =& B_1(\zeta_1) \ {}_{N-1}\langle \tilde{0} | \prod_{i=2}^{N} \beta_i(\zeta_i) | \tilde{0} \rangle_{N-1} \\
&+ B_2(\zeta_1) \sum_{j>1} c_{1,j} \ {}_{N-2}\langle \tilde{0} | \prod_{\substack{i=2 \\ i\neq j}}^{N} \beta_i^{(j)}(\zeta_i) | \tilde{0} \rangle_{N-2},
\end{aligned}$$

which has the exact same form as the Tarasov's recurrence (1.2.6).

Let us get back to (1.3.5). The integrand in (1.3.5)

$$\mathcal{G}(x) = \exp \left(x^2 \sum_{1\leq i<j\leq N} c_{i,j} B_2(\zeta_i) + x \sum_{1\leq i\leq N} B_1(\zeta_i) \right) \tag{1.3.13}$$

is the generating function of the eigenstates of the transfer matrix (1.2.6). Expression (1.3.13) is motivated by the telephone numbers t_n (sequence A000085 in OEIS). The first few telephone numbers are

$$1, 1, 2, 4, 10, 26, 76, 232, 764, 2620, 9496, ..$$

These numbers coincide with the number of terms in the polynomial Φ_N (starting from $N = 0$), which can be checked by counting terms in Φ_N using the recurrence relation (1.2.6). Indeed, if we denote the number of monomials in Φ_N in (1.2.6) by t_N, then the equation (1.2.6) says that t_N is equal to the number of monomials in Φ_{N-1}, i.e. to t_{N-1}, plus the number monomials in Φ_{N-2} taken $N-1$ times. Therefore the recurrence (1.2.6) implies the recurrence on the number of monomials t_N in Φ_N

$$t_N = t_{N-1} + (N-1)t_{N-2}.$$

This is precisely the recurrence relation for the telephone numbers.

The form of the generation function $\mathcal{G}$ (1.3.13) is a generalization of the generation function for the numbers t_n

$$g(x) = \sum_{n=0}^{\infty} t_n \frac{x^n}{n!} = \exp\left(\frac{x^2}{2} + x\right).$$

Generalizing $g(x)$ to $g(x;\mathcal{O}_1,\mathcal{O}_2)$

$$g(x;\mathcal{O}_1,\mathcal{O}_2) = \exp\left(x^2\mathcal{O}_2 + x\mathcal{O}_1\right),$$

and assuming that $\mathcal{O}_1$ and $\mathcal{O}_2$ are operators we obtain an expression similar to (1.3.13). It remains to match $\mathcal{O}_1$ and $\mathcal{O}_2$ with the coefficients of x and x^2 in the exponential (1.3.13).

Although the exponential form of the eigenstates $|\Psi_N\rangle$ (1.3.5) solves the recurrence relation (1.2.6) it remains unclear what are the precise advantages of the expression (1.3.5). We hope that the factorized form of the eigenstates written in terms of the new operators (1.3.1) will allow us to make further steps towards computing the scalar products and the form factors of the IK model similarly to the case of the six vertex model [84, 78].

1.4 Scalar products

Since we know how to construct the eigenstates of the transfer matrix we can address the problem of the computation of correlation functions. The simplest correlation functions are the form factors, expectation values of the local spin operators. Let us clarify what we mean by the expectation value. For that we need to introduce the dual Bethe states. The dual states are constructed form the dual pseudo vacuum state $\langle 0|$. This state is defined by similar properties to the state $|0\rangle$

$$\langle 0|B_i(v) = 0, \quad \langle 0|A_i(v) = \langle 0|\alpha_i(v), \quad \langle 0|C_i(v) \neq 0. \tag{1.4.1}$$

Notice that the roles of the elements B and C are reversed. The dual eigenfunctions of the transfer matrix are defined by the action of a polynomial $\bar{\Phi}_N$ in the elements of the Yang–Baxter algebra on the dual pseudo vacuum. The formula for the dual Bethe states is analogous to (1.3.4)

$$\langle\bar{\Psi}_N(\zeta_1,..,\zeta_N)| = \langle 0| \otimes \langle\tilde{0}| \prod_{i=1}^{N} \gamma(\zeta_i|\zeta_{i+1},..,\zeta_N)|\tilde{0}\rangle, \tag{1.4.2}$$

where γ is

$$\gamma(\zeta_i|\zeta_{i+1},..,\zeta_N) = \mathbb{I} + C_1(\zeta_i)f_i + C_2(\zeta_i) \times \sum_{j>i} \tilde{c}_{i,j} f_j f_i, \quad (i = 1,..N), \tag{1.4.3}$$

where all ingredients were defined earlier except for the coefficients $\tilde{c}_{i,j}$. These coefficients are, in fact, the same as $c_{i,j}$ but with the weights x_6 and y_6 interchanged. The discussion of the generating functions for the Bethe states in Section 1.3 is straightforwardly translated to the dual Bethe states. In particular, we have

$$\langle \bar{\Psi}_N| = \oint \frac{\mathrm{d}x}{x^{N+1}} \langle 0| \colon \exp\left(x^2 \sum_{1\le i<j\le N} \tilde{c}_{i,j} C_2(\zeta_i) + x \sum_{1\le i\le N} C_1(\zeta_i)\right) \colon , \tag{1.4.4}$$

or

$$\begin{aligned} \langle \bar{\Psi}_N| &= \oint \frac{\mathrm{d}x}{x^{N+1}} \langle 0| \colon \prod_{i=1}^{N} e^{x\mathcal{C}(\zeta_i;x)} \colon , \\ \mathcal{C}(\zeta_i;x) &= xC_2(\zeta_i) \sum_{i<j\le N} \tilde{c}_{i,j} + C_1(\zeta_i). \end{aligned}$$

The scalar products of states are defined as

$$S_N(\mu_1,..,\mu_N;\zeta_1,..,\zeta_N) = \langle \bar{\Psi}_N(\mu_1,..,\mu_N)|\Psi_N(\zeta_1,..,\zeta_N)\rangle. \tag{1.4.5}$$

Thus we can write S_N as

$$S_N = \oint \frac{\mathrm{d}x\mathrm{d}y}{x^{N+1}y^{N+1}} \langle 0| \colon \prod_{i=1}^{N} e^{x\mathcal{C}(\mu_i;x)} \colon \colon \prod_{i=1}^{N} e^{y\mathcal{B}(\zeta_i;y)} \colon |0\rangle.$$

If $\zeta_1,..,\zeta_N$ are Bethe roots and $\mu_1,..,\mu_N$ are also Bethe roots, then the quantity in (1.4.5) becomes the normalisation of the N-particle state. One is usually interested in the case when one of the two sets of parameters are Bethe roots while the other one is free. If the parameters μ are kept free then S_N is called the off-shell on-shell scalar product, we will simply call it the scalar product. Indeed, if we want to compute the expectation value of an operator $\mathcal{O}$

$$\langle \mathcal{O}\rangle = \langle \bar{\Psi}_N(\mu_1,..,\mu_N)|\mathcal{O}|\Psi_N(\zeta_1,..,\zeta_N)\rangle, \tag{1.4.6}$$

and, say, we computed the action of $\mathcal{O}$ on the dual Bethe state written as a combination of dual states

$$\langle \bar{\Psi}_N(\mu_1,..,\mu_N)|\mathcal{O} = \sum_k \theta_k \langle \bar{\Psi}_N(\nu_1^{(k)},..,\nu_N^{(k)})|, \tag{1.4.7}$$

where $\nu_i^{(k)}$ are some new numbers now. The computation of $\langle \mathcal{O}\rangle$ reduces to the computation of the scalar product (1.4.5). Since the states $|\Psi_N\rangle$ and $\langle \bar{\Psi}_N|$ have a complicated form, it remains unclear for now how to show the validity of the expansion (1.4.7). It is also unclear how to express, say, local operators in terms of the Yang–Baxter algebra. The above discussion of the expectation value of an operator $\mathcal{O}$ is inspired by the XXZ spin chain or the six vertex model ($U_q(\hat{sl}_2)$), where these issues are well understood. There, one [111, 112] finds nice formulae for the scalar product S_n. The big distinction between the case of $U_q(\hat{sl}_2)$ and our $U_q(A_2^{(2)})$ is that in the latter the Bethe states have very complicated form. The Yang–Baxter algebra of the Tarasov's algebraic Bethe Ansatz is not easy to work with and for now it is unclear how to proceed. Possibly, the new presentations of the eigenstates of the transfer matrix written as exponentials (1.3.5) and (1.4.4) can be helpful. However, it is far from obvious at the moment how to make use of it.

We give a final remark of this section. The calculation of scalar products for $U_q(\hat{sl}_2)$ [84, 78] relies on the so called Izergin–Korepin partition function. This partition function is extremely important in many other respects and, in particular, it is intimately related to the scalar products. It is natural to ask what is the analog of the Izergin–Korepin partition function for the model we consider here. The Izergin–Korepin partition function, otherwise called the partition function of the domain wall six vertex model (DWPF), is the sum of all configuration of the six vertex model on a square domain with the boundaries fixed as in Fig. 1.8. The naive translation of this into

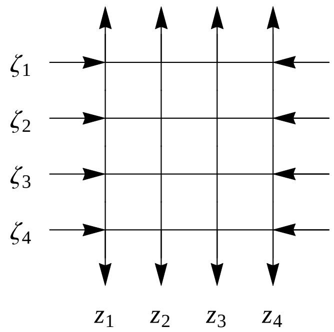


FIGURE 1.8 – The domain wall boundary conditions for 4×4 lattice.

the case of the IK nineteen vertex model is probably not the correct generalization of DWPF once the calculation of the scalar products is concerned. However, it is probably the first step towards the right object. In Chapter 2 we address this problem.

Chapter 2

The domain wall partition function for the IK model

In this chapter we study a particular object of the nineteen vertex model of the $U_q(A_2^{(2)})$ quantum group. The object of our interest is the partition function of the model on a square lattice in a $N \times N$ square region with the domain wall boundary conditions [1].

As mentioned previously, we are motivated by the domain wall partition function (DWPF) for the six vertex model Z^{6v}, constructed using the R-matrix of the $U_q(A_1^{(1)})$ ($U_q(\hat{sl}_2)$) quantum group. Korepin [83] obtained a set of recurrence relations for Z^{6v} which were solved by Izergin [65] and thus this partition function is called the Izergin–Korepin (IK) partition function. In statistical physics the six vertex model represents a model for two dimensional ice, which shows interesting critical phenomena (see [2]). The partition function Z^{6v} plays a very important role in the field of integrable models. It is a crucial object in the theory of correlation functions for integrable spin chains [84] such as the XXZ spin-1/2 chain (see also [78]). In combinatorics it allowed the counting of alternating sign matrices and their symmetry classes [89]. To compute the domain wall partition functions for other vertex model is a very complicated problem. One of the main results generalizing the six vertex domain wall partition function (DWPF) is due to [20], where the $U_q(A_1^{(1)})$ higher spin generalization of the DWPF is obtained in a determinant form. Inspired by this, we address the question of computing the domain wall partition function for the $U_q(A_2^{(2)})$ nineteen vertex model.

The IK R-matrix has nineteen non zero entries (1.1.3), which correspond to the nineteen possible vertex configurations (see Fig. 2.5 and also Fig. 1.3). We use this R-matrix to build N by N lattice configurations which have the domain wall boundary conditions Fig. 2.2. The sum of all such configurations we call the domain wall partition function Z_N. In order to compute Z_N we use the ideas from the six vertex model. First, we establish the recurrence relation for the partition function and then try to find its unique solution. In the case when the deformation parameter q is generic we cannot find a compact expression for Z_N. However, when $q^3 = -1$ we are able to find a determinant expression.

In Section 2.1 we briefly discuss the DWPF for the six vertex model. In Section 2.2

1. The material of this chapter is the subject of the preprint [55] of the author.

we move to the IK model. In Section 2.3 we derive the recurrence relation using the vanishing properties of the weights of the R-matrix. The solution to this recurrence relation at $q^3 = -1$ is presented in Section 2.4. The proof is given in Section 2.5. We complete this chapter with a summary of results.

2.1 Six-vertex model with domain wall boundary

For the computation of the IK determinant for the six vertex model we refer to the papers [65, 83]. Here we present a short discussion for convenience.

The problem is counting the number of configurations which are built by choosing for each vertex of a square $N \times N$ lattice one of the six vertices from Fig. 2.1. A

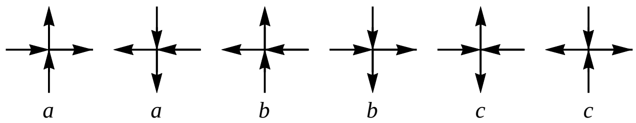


FIGURE 2.1 – The six vertices of the six vertex model. The letters a, b, and c are the weights of the corresponding vertices.

configuration thus constructed will have on each edge one of the two states: a left arrow or a right arrow if the edge is horizontal and an up arrow or a down arrow for a vertical edge. We then impose the domain wall boundary conditions as on the example shown in Fig. 2.2. Each vertex on this lattice has a position (i,j) where horizontal

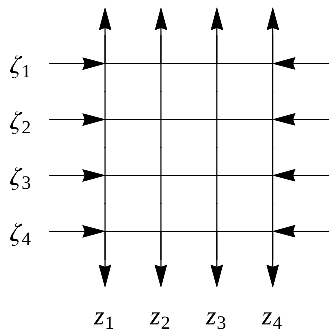


FIGURE 2.2 – The domain wall boundary conditions on a 4×4 lattice. The parameters ζ_1,..,ζ_4 are associated to the horizontal lines, while the parameters z_1,..,z_4 are associated to the vertical lines.

position i is counted rightwards, and the vertical position j is counted downwards starting from the top left corner. The weight of the vertex at the position (i,j) is denoted by $w_{i,j}$ and takes one of the three values $a_{i,j}$, $b_{i,j}$ or $c_{i,j}$. The weight of a configuration ε on a square domain of size $N \times N$ will be the product of all weights of its vertices

$$\prod_{1 \le i,j \le L} w_{i,j}^{(\varepsilon)}.$$

The IK partition function is the sum over all configurations (states) ε

$$Z^{6v} = \sum_{\varepsilon \in \text{states}} \prod_{1 \le i,j \le N} w_{i,j}^{(\varepsilon)}. \tag{2.1.1}$$

The weights a, b and c are encoded in the R-matrix [2]. The R-matrix acts on two vector spaces labeled by i and j, which carry spectral parameters z_i and z_j, thus we write $R_{i,j}(z_i,z_j)$. We write the R-matrix in the spin basis: $e_+ = (1,0)$ and $e_- = (0,1)$, where e_+ corresponds to an up arrow if the edge is vertical and a right arrow if the edge is horizontal, similarly e_- corresponds to a down arrow if the edge is vertical and a left arrow if the edge is horizontal.

$$R_{i,j}(z_i,z_j) = \begin{pmatrix} a(z_i,z_j) & 0 & 0 & 0 \\ 0 & b(z_i,z_j) & c(z_i,z_j) & 0 \\ 0 & c(z_i,z_j) & b(z_i,z_j) & 0 \\ 0 & 0 & 0 & a(z_i,z_j) \end{pmatrix} \tag{2.1.2}$$

In fact the integrable R-matrix depends on the ratio of the spectral parameters: $R_{i,j}(z_i/z_j) \propto R_{i,j}(z_i,z_j)$. Using the matrix units $\epsilon_{a,b}$ as the basis for the matrices acting in $\mathbb{C}^2$ we can write as before (1.1.7) (the indices in the summations in this section take values $-$ and $+$)

$$R(z_1/z_2) = \sum_{a,b,c,d} r_{a,b}^{c,d}(z_1/z_2)\epsilon_{a,c} \otimes \epsilon_{b,d}, \tag{2.1.3}$$

where the components of the R-matrix are denoted by $r_{a,b}^{c,d}$; furthermore we will again use their graphical representation as in Fig. 1.1 of Chapter 1. We will also need the $\check{R}$-matrix: $\check{R} = PR$, where P is the permutation matrix now acting in the tensor of two copies of $\mathbb{C}^2$

$$P = \sum_{a,b} \epsilon_{a,b} \otimes \epsilon_{b,a}, \tag{2.1.4}$$

so we have

$$\check{R}(z_1/z_2) = \sum_{a,b,c,d} \check{r}_{a,b}^{c,d}(z_1/z_2)\epsilon_{a,c} \otimes \epsilon_{b,d}. \tag{2.1.5}$$

Graphically, the components of $\check{R}$ are presented in Fig. 1.2. The integrable R-matrix satisfies the Yang–Baxter equation. Using the schematic notation of the $\check{R}$-matrix this equation can be drawn as in Fig. 2.3. The Yang–Baxter equation corresponding to Fig. 2.3 is written then as

$$\check{R}_{i+1}(y,x)\check{R}_i(z,x)\check{R}_{i+1}(z,y) = \check{R}_i(z,y)\check{R}_{i+1}(z,x)\check{R}_i(y,x), \tag{2.1.6}$$

where $\check{R}$-matrices here are: $\check{R}_i = \check{R} \otimes Id$ and $\check{R}_{i+1} = Id \otimes \check{R}$. This equation restricts the possible weighs of the vertices. The solution reads

$$a(z_i,z_j) = \frac{q^2 z_i^2 - z_j^2}{(q^2-1)\, z_i z_j}, \quad b(z_i,z_j) = \frac{q\left(z_i^2 - z_j^2\right)}{(q^2-1)\, z_i z_j}, \quad c(z_i,z_j) = 1. \tag{2.1.7}$$

2. In this section R is used to denote the six vertex R-matrix.

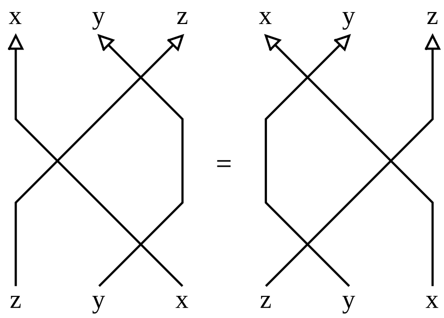


FIGURE 2.3 – The Yang–Baxter equation. The spectral parameters x, y and z are carried by the corresponding vector spaces.

In the square lattice domain as on Fig. 2.2 there are N horizontal spaces carrying N parameters $\zeta_1,..,\zeta_N$ and N vertical spaces carrying N parameters $z_1,..,z_N$. The latter parameters are called inhomogeneities and the model therefore is called the inhomogeneous six vertex model. From the form of the weights Eq. (2.1.7) we see that the partition function Z^{6v} is a polynomial in z's and ζ's divided by a common denominator that we neglect in what follows. In fact, Z^{6v} is symmetric separately in z's and in ζ's. It can be seen by applying the $\check{R}_{i,i+1}$ matrix to Fig. 2.2 and using repeatedly the Yang–Baxter equation. If the $\check{R}$-matrix is applied at a position i from below or above of the domain Fig. 2.2 this action will switch two rapidities z_i and z_{i+1}, if it is applied from the sides ζ_i and ζ_{i+1} will be switched (see Fig. 2.4).

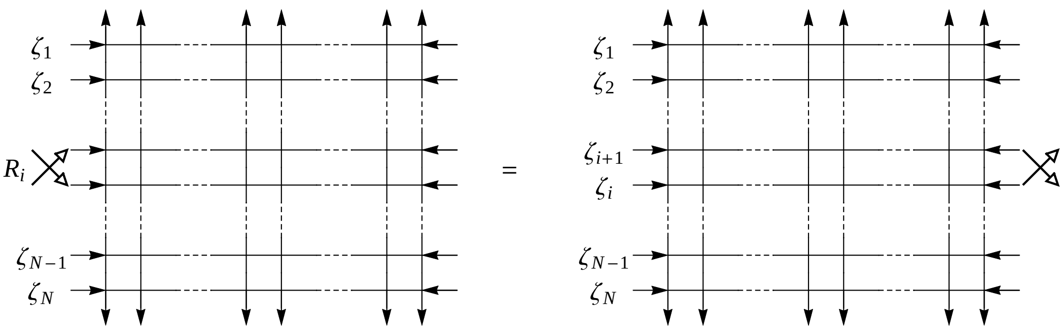


FIGURE 2.4 – On the left side of this equation one must use the Yang–Baxter equation Fig. 2.3 to push through the $\check{R}$-matrix. The boundary conditions are such that on the both sides of this equation there is only one term of the $\check{R}$-matrix that contributes, i.e. the vertex with the weight a. The symmetry in z's is proven similarly.

Now we present the computation of the domain wall partition function Z_N^{6v} for the 6-vertex model. Z_N^{6v} has two recurrence relations that correspond to setting $\zeta_j = z_i$ and $\zeta_j = q^{-1}z_i$. For their derivation one can consult [65, 83] or see the explanation of similar recurrences in the case of the nineteen vertex model in Section 2.3. The recurrence relations are

$$Z_N^{6v}(\zeta_1,..,\zeta_j = z_i,..,\zeta_N|z_1,..,z_N) = f_{i,j}^N Z_{N-1}[\zeta_j,z_i], \tag{2.1.8}$$

and

$$Z_N^{6v}(\zeta_1,..,\zeta_j = q^{-1}z_i,..,\zeta_N|z_1,..,z_N) = g_{i,j}^N Z_{N-1}[\zeta_j,z_i], \tag{2.1.9}$$

where the square brackets indicate which variables are absent from the initial list of variables on the left hand side. The corresponding factors in the two recurrences are

$$f_{i,j}^N = \prod_{1\le k\ne i\le N}(q^2z_i^2 - z_k^2)\prod_{1\le k\ne j\le N}(q^2\zeta_k^2 - z_i^2), \tag{2.1.10}$$

$$g_{i,j}^N = \prod_{1\le k\ne i\le N}(z_i^2 - q^2z_k^2)\prod_{1\le k\ne j\le N}(\zeta_k^2 - z_i^2). \tag{2.1.11}$$

Then Z_N^{6v} is expressed in terms of Z_{N-1}^{6v}'s as

$$Z_N^{6v}(\zeta_1,..,\zeta_N|z_1,..,z_N) = \sum_{k=1}^N Z_{N-1}[\zeta_N,z_k]\prod_{i=1,i\ne k}^N \frac{(\zeta_N^2 - z_i^2)}{(z_k^2 - z_i^2)} f_{i,N}^N, \quad \text{and} \tag{2.1.12}$$

$$Z_N^{6v}(\zeta_1,..,\zeta_N|z_1,..,z_N) = \sum_{k=1}^N Z_{N-1}[\zeta_N,z_k]\prod_{i=1,i\ne k}^N \frac{(q^2\zeta_N^2 - z_i^2)}{(z_k^2 - z_i^2)} g_{i,N}^N. \tag{2.1.13}$$

These recurrence relations were established by Korepin and solved by Izergin and the solution is written as the following determinant

$$Z_N^{6v} = \mathcal{N}\det_{1\le i,j\le N}\left(\frac{1}{(\zeta_i^2 - z_j^2)(q^2\zeta_i^2 - z_j^2)}\right), \tag{2.1.14}$$

$$\mathcal{N} = \frac{\prod_{1\le i,j\le N}(\zeta_i^2 - z_j^2)(q^2\zeta_i^2 - z_j^2)}{\prod_{1\le i<j\le N}(\zeta_i^2 - \zeta_j^2)(z_j^2 - z_i^2)\prod_{i=1}^N(q^2-1)^{N-1}\zeta_i^{N-1}z_i^{N-1}}.$$

Up to a denominator which contains the product of $\zeta_i^{N-1}z_i^{N-1}$, the partition function Z_N^{6v} is a polynomial of degree $(N-1)$ in each variable ζ_i^2 and z_i^2 and it satisfies the required recurrence relations together with the initial condition $Z_1^{6v} = 1$.

2.2 Nineteen-vertex model with domain wall boundary

Consider an inhomogeneous nineteen-vertex model on a lattice. States of the model are defined through assigning one of the nineteen configurations to each vertex of the lattice. Each edge of the lattice can be in three states, denoted by arrows or an empty edge, in such a way that the total number of arrows pointing towards a vertex has to be equal to the total number of arrows pointing outwards. This restriction defines the nineteen possible configurations at each vertex Fig. 2.5.

The weights of the nineteen vertices are encoded in the R-matrix (1.1.3). For the

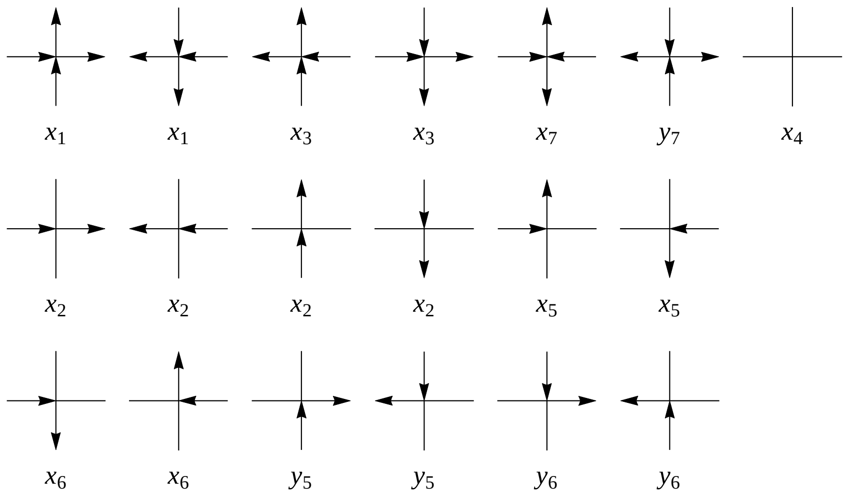


FIGURE 2.5 – The nineteen vertices and their weights.

purpose of this chapter we write the weights (1.1.4) in the multiplicative convention

$$
\begin{aligned}
x_1(\zeta) &= \left(\zeta q^2 - 1\right)\left(\zeta q^3 + 1\right), \\
x_2(\zeta) &= (\zeta - 1)\, q \left(\zeta q^3 + 1\right), \\
x_3(\zeta) &= (\zeta - 1)\, q^2 \left(\zeta q + 1\right), \\
x_4(\zeta) &= -\zeta + \zeta q^5 + \zeta\,(\zeta - 1)\, q^4 - \zeta q^3 + \zeta q^2 + (\zeta - 1)\, q, \\
x_5(\zeta) &= \sqrt{\zeta}\left(q^2 - 1\right)\left(\zeta q^3 + 1\right), \\
x_6(\zeta) &= \sqrt{\zeta}\,(\zeta - 1)\,(-\sqrt{q})\left(q^2 - 1\right), \\
x_7(\zeta) &= \zeta\left(q^2 - 1\right)\left(\zeta q^3 + (\zeta - 1)\, q + 1\right), \\
y_5(\zeta) &= \sqrt{\zeta}\left(q^2 - 1\right)\left(\zeta q^3 + 1\right), \\
y_6(\zeta) &= \sqrt{\zeta}\,(\zeta - 1)\, q^{5/2}\left(q^2 - 1\right), \\
y_7(\zeta) &= \left(q^2 - 1\right)\left(\zeta q^3 - (\zeta - 1)\, q^2 + 1\right).
\end{aligned}
\tag{2.2.1}
$$

We are interested in counting configurations of the following object. Consider the square lattice of size N filled in with the above nineteen vertices in such a way that the horizontal boundary arrows are pointing in to the lattice, while the vertical ones pointing outside the lattice. These boundary conditions are presented in Fig. 2.2. The corresponding partition function is the sum of all possible configurations with weights defined in Eq. (2.2.1)

$$
Z_N = \sum_{\varepsilon \in \text{states}} \prod_{1 \le i,j \le N} w_{i,j}^{(\varepsilon)}, \tag{2.2.2}
$$

where $w_{i,j}^{(\varepsilon)}$ is the weight of the vertex sitting at the position (i,j) of a configuration ε. This partition function is a symmetric polynomial in both horizontal ζ_i and vertical z_i

rapidities. The fact that it is a polynomial comes from the observation that each vertex that has a $\sqrt{\zeta}$ appears necessarily with another vertex that has a $\sqrt{\zeta}$. These weights are: x_5, x_6, y_5 and y_6 and they correspond to the vertices which have a "turning" of an empty line. Clearly, the number of such turnings must be even in any DWPF configuration. The fact that Z_N is symmetric can be proved as in the case of the six vertex model by attaching the R-matrix to two horizontal external lines of Fig. 2.2 or two vertical external lines and repetitive application of the Yang–Baxter equation. Hence the partition function $Z_N(\zeta_1,..,\zeta_N,z_1,..,z_N)$ is a symmetric polynomial in z_i's and ζ_j's with coefficients being polynomials in q with integer coefficients.

2.3 Recurrence relation

The partition function Z_N satisfies two recurrence relations in size with the initial condition $Z_0 = 1$. They both take the form

$$Z_N(\zeta_1,..,\zeta_N|z_1,..,z_N) = \sum_{i=1}^{N} \kappa_i(\zeta_1,..,\zeta_N|z_1,..,z_N) Z_{N-1}(\zeta_1,..,\zeta_{N-1}|..,\hat{z}_i,..), \tag{2.3.1}$$

with some appropriate polynomials κ_i.

By inspecting the vanishing properties of the weights of the R-matrix we notice that there are two recurrence relations in size. When we set ζ_j to z_i in Z_n we get

$$Z_N(\zeta_1,..,\zeta_j = z_i,..,\zeta_N|z_1,..,z_N) = F^N_{i,j} Z_{N-1}[\zeta_j,z_i], \tag{2.3.2}$$

This recurrence has a graphical interpretation shown in Fig. 2.6. Indeed, if we look at the north east corner (position $(1,N)$ on the lattice) of the domain, the boundary condition allows only for three vertices. These are vertices with the weights x_3, x_7 and x_6. After setting $\zeta_1 = z_N$, x_3 and x_6 vanish, so we are left with the vertex x_7. This vertex has a down arrow on its vertical lower edge and a right arrow on its left edge, hence due to the boundary condition at the position $(2,N)$ we are forced to put there the vertex corresponding to the weight x_1 and at the position $(1,N-1)$ the other vertex with the weight x_1. In fact, all remaining vertices in the N-th column are frozen, as well as all the remaining vertices of the first row. These vertices contribute with the products of x_1-weights

$$\prod_{1\le i\le N-1} x_1(\zeta_1/z_i) \prod_{2\le i\le N} x_1(\zeta_i/z_N)|_{\zeta_1=z_N}. \tag{2.3.3}$$

A different recurrence appears when we set ζ_j to $-q^{-3}z_i$ in Z_N

$$Z_N(\zeta_1,..,\zeta_j = -q^{-3}z_i,..,\zeta_N|z_1,..,z_N) = G^N_{i,j} Z_{N-1}[\zeta_j,z_i], \tag{2.3.4}$$

The graphical explanation of this recurrence is similar to the previous recurrence. One must consider the top left corner of our domain and observe that only one vertex does not vanish under the substitution $\zeta_1 = -q^{-3}z_1$. The first row and the first column freeze, while the rest returns the domain wall boundary condition for the domain of the size $(N-1)\times(N-1)$.

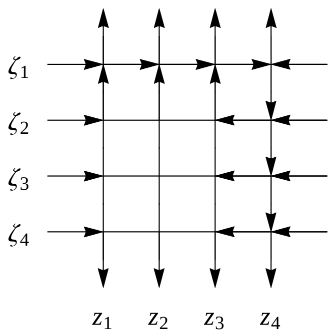


FIGURE 2.6 – The recurrence relation under substitution $\zeta_1 = z_4$ for a 4×4 lattice. Since the first row is frozen and the last column is frozen we obtain simple factors of x_1-weights, while the remaining configuration has the domain wall boundary conditions and corresponds to Z_3.

The F and the G are given by

$$F^N_{i,j} = (q^3+1)z_i \prod_{1\le k\neq i\le N}(q^2 z_i - z_k)(q^3 z_i + z_k)\prod_{1\le k\neq j\le N}(q^2\zeta_k - z_i)(q^3\zeta_k + z_i), \quad (2.3.5)$$

$$G^N_{i,j} = -q^{-N-1}(q^3+1)z_i \prod_{1\le k\neq i\le N}(z_i - q^2 z_k)(z_i + q^3 z_k)\prod_{1\le k\neq j\le N}(\zeta_k - z_i)(q\zeta_k + z_i). \quad (2.3.6)$$

If we know Z_{N-1} these two recurrence relations allow us to determine Z_N. We can consider Z_N as a polynomial in ζ_N of degree $2N-1$ with $2N$ coefficients. Since we know the values of Z_N at N points $\zeta_N = z_i$ (Eq. (2.3.2) with $j = N$) and at another N points $\zeta_N = -q^{-3}z_i$ (Eq. (2.3.4) with $j = N$), therefore we can determine all the coefficients of Z_N in its expansion in ζ_N. Using the Lagrange polynomial we can write Z_N as a sum of Z_{N-1}'s as follows

$$Z_N(\zeta_1,..,\zeta_N|z_1,..,z_N) = \sum_{k=1}^N Z_{N-1}[\zeta_N, z_k]\frac{\prod_{i=1}^N(\zeta_N - z_i)(\zeta_N + q^{-3}z_i)}{\prod_{i=1,i\neq k}^N(z_k - z_i)}.$$
$$\times\left(\frac{F^N_{k,N}}{(\zeta_N - z_k)\prod_{i=1}^N(z_k + q^{-3}z_i)} - \frac{q^{3(N-1)}G^N_{k,N}}{(\zeta_N + q^{-3}z_k)\prod_{i=1}^N(q^{-3}z_k + z_i)}\right). \quad (2.3.7)$$

This is of course a polynomial because the denominators are canceled by the common prefactor and by the F and G respectively including the factors of $q^3 - 1$ in the definitions Eq. (2.3.5) and Eq. (2.3.6). Using this we write

$$Z_N(\zeta_1,..,\zeta_N|z_1,..,z_N) = q^{3(N-1)}\sum_{k=1}^N Z_{N-1}[\zeta_N, z_k]\prod_{i=1,i\neq k}^N\frac{(\zeta_N - z_i)(\zeta_N + q^{-3}z_i)}{(z_k - z_i)}$$
$$\times\Bigg((q^3\zeta_N + z_k)\prod_{i\neq k}(q^2 z_k - z_i)\prod_{1\le i\le N-1}(-z_k + q^2\zeta_i)(z_k + q^3\zeta_i)$$
$$+ q^{2N-1}(\zeta_N - z_k)\prod_{i\neq k}(z_k - q^2 z_i)\prod_{1\le i\le N-1}(-z_k + \zeta_i)(z_k + q\zeta_i)\Bigg). \quad (2.3.8)$$

Possibly there is a way to write Z_N for generic q as a single determinant, for now this remains an open question. In Section 2.4 we show how to solve the recurrence relation for Z_N when $q^3 = -1$.

2.4 Solution for the cubic root of unity

In this section we will assume $q^3 = -1$. The recurrence relation Eq. (2.3.8) simplifies in this case. Upon setting $q^3 = -1$ we observe from Eq. (2.3.8) that Z_N factors out the product

$$\prod_{1\le i,j\le N} (z_i - \zeta_j), \tag{2.4.1}$$

which we neglect in the following. The initial condition becomes $Z_1 = 1$ and out of the two recurrence points only one remains, i.e. when we set $\zeta_j = z_i$

$$Z_N(\zeta_1,..,\zeta_j = z_i,..,\zeta_N|z_1,..,z_N) = P_{i,j} Z_{N-1}[\zeta_j,z_i]. \tag{2.4.2}$$

Let us focus on the polynomial $P_{i,j}$, and for convenience we specify $i = N$, $j = N$ and set $z_N = x$. The polynomial $P_{N,N} = P(x|\zeta_1,..,\zeta_{N-1},z_1,..,z_{N-1})$ is a symmetric polynomial in $\zeta_1,..,\zeta_{N-1}$ and separately in $z_1,..,z_{N-1}$

$$P(x|\zeta_1,..,\zeta_{N-1},z_1,..,z_{N-1}) =$$
$$(-q)^N \left(q \prod_{i=1}^{N-1} (\zeta_i + qx) \prod_{i=1}^{N-1} (z_i + x/q) + \frac{1}{q} \prod_{i=1}^{N-1} (\zeta_i + x/q) \prod_{i=1}^{N-1} (z_i + qx) \right). \tag{2.4.3}$$

Note up to the overall factor of q^N, P is invariant under $q \to 1/q$, which means it has to be a function of $q + 1/q$. Since we set $q^3 = -1$ we have $q + 1/q = 1$ and P becomes a polynomial with purely integer coefficients. The same is therefore also true for the Z_N itself. Let us consider now P as the generating function for some symmetric polynomials

$$P_N(x) = P(x|\zeta_1,..,\zeta_N,z_1,..,z_N) = (-q)^N \sum_{i=0}^{2N} x^i \Delta_{2N-i,N}(\zeta_1,..,\zeta_N,z_1,..,z_N). \tag{2.4.4}$$

We included here the factor of q^N in order to make $\Delta_{i,N}$ q-independent. The polynomials $\Delta_{i,N}$ are polynomials of $2N$ variables with the total degree i. If $i < 0$ or $i > 2N$ we set it equal to 0, and also $\Delta_{0,N} = 1$. Here is the example for $N = 2$

$$\begin{aligned}
\Delta_{1,2} &= 2\zeta_1 + 2\zeta_2 - z_1 - z_2,\\
\Delta_{2,2} &= \zeta_1\zeta_2 + \zeta_1 z_2 + \zeta_2 z_2 + \zeta_1 z_1 + \zeta_2 z_1 - 2 z_1 z_2,\\
\Delta_{3,2} &= -\zeta_1 z_1 z_2 + 2\zeta_2\zeta_1 z_1 + 2\zeta_2\zeta_1 z_2 - \zeta_2 z_1 z_2,\\
\Delta_{4,2} &= \zeta_1\zeta_2 z_1 z_2.
\end{aligned}$$

These symmetric functions have a few nice properties which we will discuss in Section 2.5. The solution to the recurrence relation (2.4.2) reads

$$Z_N(\zeta_1,..,\zeta_N,z_1,..,z_N) = \det_{1\le i,j\le N-1} \Delta_{3j-i,N}(\zeta_1,..,\zeta_N,z_1,..,z_N), \tag{2.4.5}$$

This is the main result that we present in this chapter. The proof of this formula follows next.

2.5 Proof

Let us list few properties of $\Delta_{i,N}$. First of all, looking at the definition of these polynomials we can immediately express them through the elementary symmetric polynomials

$$\Delta_{2N-i,N} = \sum_{\substack{0\leq n_1,n_2\leq N \\ n_1+n_2=i}}^{N} (q^{1-n_1+n_2} + q^{-1+n_1-n_2}) E_{N-n_1}(\zeta_1,..,\zeta_N) E_{N-n_2}(z_1,..,z_N), \tag{2.5.1}$$

where $E_i(z_1,..,z_N)$ is 0 if $i<0$ or $i>N$, otherwise

$$E_i(z_1,..,z_N) = \sum_{1\leq n_1<..<n_i\leq N} z_{n_1} z_{n_2}..z_{n_i}. \tag{2.5.2}$$

Note that Eq. (2.5.1) is valid for generic values of q. When $q=1$, Δ_i become the elementary symmetric polynomials of the union of z's and ζ's times a factor of two. So, it can be considered as a type of q-deformation of the elementary symmetric polynomials.

Let us look at what happens when we set, say, $\zeta_N = z_N$. From the definition of P_N we see that it produces back P_{N-1}

$$\begin{aligned} &P(x|\zeta_1,..,\zeta_N,z_1,..,z_N)|_{\zeta_N=z_N} = -(z_N q + x)(z_N + qx) P(x|\zeta_1,..,\zeta_{N-1},z_1,..,z_{N-1}) \\ &\quad = -q(z_N^2 + x z_N + x^2) P(x|\zeta_1,..,\zeta_{N-1},z_1,..,z_{N-1}), \end{aligned} \tag{2.5.3}$$

where in the second line we took into account that $q^3=-1$. Looking at Eq. (2.5.3) we can relate the set of $\Delta_{i,N}$'s in which $\zeta_N = z_N$ to the set of $\Delta_{j,N-1}$'s

$$\begin{aligned} &\Delta_{i,N}(\zeta_1,..,\zeta_N = z_N,z_1,..,z_N) = \Delta_{i,N-1}(\zeta_1,..,\zeta_{N-1},z_1,..,z_{N-1}) \\ &+ z_N \Delta_{i-1,N-1}(\zeta_1,..,\zeta_{N-1},z_1,..,z_{N-1}) + z_N^2 \Delta_{i-2,N-1}(\zeta_1,..,\zeta_{N-1},z_1,..,z_{N-1}). \end{aligned} \tag{2.5.4}$$

Using this equation and a certain row-column manipulation in the matrix $\Delta_{3j-i,N}$ we are going to show that the determinant (2.4.5) satisfies the recurrence (2.4.2).

Set $\zeta_N = z_N$ and substitute Eq. (2.5.4) in every entry of the matrix in Eq. (2.4.5). Starting from the first row subtract from each row i row $i+1$ multiplied by z_N. Next, subtract from each column j column $j+1$ multiplied by $z_N^{3(N-1-j)}$ starting from the $j=(N-2)$-th column. In the resulting matrix all elements of the first column become zero except from the bottom element. The bottom element in the first column takes the form of Eq. (2.4.4), while the rest of the matrix equals to $\Delta_{3j-i,N}$ of size $N-1$, and the last row is unimportant upon taking the determinant. The row-column manipulation above corresponds to the following series of equations. Application of the recurrence relation in each matrix entry gives

$$\Delta_{3j-i,N-1} + z_N \Delta_{3j-i-1,N-1} + z_N^2 \Delta_{3j-i-2,N-1} \tag{2.5.5}$$

After the first row manipulation the last row remains as before

$$\Delta_{3j-N+1,N-1} + z_N \Delta_{3j-N,N-1} + z_N^2 \Delta_{3j-N-1,N-1}, \tag{2.5.6}$$

the rest part of the matrix becomes

$$\Delta_{3j-i,N-1} - z_N^3 \Delta_{3j-i-3,N-1}. \tag{2.5.7}$$

We notice that in the last column the first of these two terms vanishes $\Delta_{3(N-1)-i,N-1}$ for all $i < N-1$, while in the first column the second term vanishes. Next, we use the last column to eliminate the unwanted terms in other entries of the matrix (except from the last row). After this, the first column except from its last element will vanish, while the last element will be

$$\begin{aligned}&\sum_{j=1}^{N-1} z_N^{3(N-1-j)}(\Delta_{3j-N+1,N-1} + z_N \Delta_{3j-N,N-1} + z_N^2 \Delta_{3j-N-1,N-1})\\ &= q^{2(N-1)} P(z_N|\zeta_1,..,\zeta_{N-1},z_1,..,z_{N-1})\end{aligned} \tag{2.5.8}$$

This completes the proof. We can alternatively view this row column manipulation as acting on the left and on the right of Eq. (2.5.5) with certain matrices with unit determinant. Let us call the expression in Eq. (2.5.5) $\tilde{\Delta}_{3j-k,N-1}$, and define two matrices

$$A = \begin{pmatrix} 1 & -z & 0 & \dots & 0 \\ 0 & 1 & -z & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & -z \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & z^3 & z^6 & \dots & z^{3(N-1)} \\ 0 & 1 & z^3 & \dots & z^{3(N-2)} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & z^3 \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}.$$

Then we have

$$\det_{1\le k,l\le N-1} A_{i,j} \tilde{\Delta}_{3j-k,N-1} B_{k,l} = q^{2(N-1)} P_{N-1}(z_N) \det_{1\le k,l\le N-2} \Delta_{3j-k,N-1}. \tag{2.5.9}$$

2.6 Discussion

As we mentioned in the introduction, our study is motivated by the six vertex model. Hence, it is natural to look at other related objects which were computed for the six vertex model. Since the nineteen vertex model seems to have a more complicated structure, one probably should not expect to obtain nice answers as in the six vertex case. As we have observed, however, when q is a root of unity the nineteen vertex model becomes "*computable*".

Here we considered the domain wall boundary conditions for the nineteen vertex model of Izergin and Korepin. An interesting extension of our computation would be to consider other boundary conditions, i.e. to use reflection matrices on one or two sides of the $N \times N$ domain. In the case of the six vertex model the corresponding partition functions are known to be determinants or Pfaffians (see [118] and also [90]). One would need to find first the recurrence relation for the partition function and then after setting $q^3 = -1$ it should be possible to obtain a determinantal expression. We note here that similar determinants appear in the study of the related loop model exactly when $q^3 = -1$. The loop model related to the IK model is called the dilute Temperley–Lieb (dTL) O(n) loop model [103, 102]. This model has a parameter, the weight n of a loop. When $q^3 = -1$ this corresponds to $n = 1$, and the corresponding loop model is related to interesting statistical models like critical percolation for example. In [58] it was shown that the sum rule of the dTL O(1) model satisfies a similar recurrence as in Eq. (2.4.2) and Eq. (2.4.3), and has a solution similar to Eq. (2.4.5).

In the context of the algebraic Bethe Ansatz the domain wall partition function for the six vertex model represents the highest spin eigenvector of the corresponding transfer matrix with periodic boundary conditions. The parameters ζ_i become the Bethe roots. This object is essential in the study of correlation functions of the corresponding model. One may similarly look at the highest spin eigenvector of the transfer matrix for the IK model. However, as we saw in Chapter 1 the eigenvectors of the transfer matrix for the nineteen vertex model are much more complicated than in the case of the six vertex model. For example, to compute the highest spin eigenvector we need to consider the nineteen vertex model with many different boundary conditions on rectangular domains. The expression for this eigenvector for $N = 4$ pictorially is shown in Fig. 2.7. In general, the expression for this eigenvector looks very complicated. For

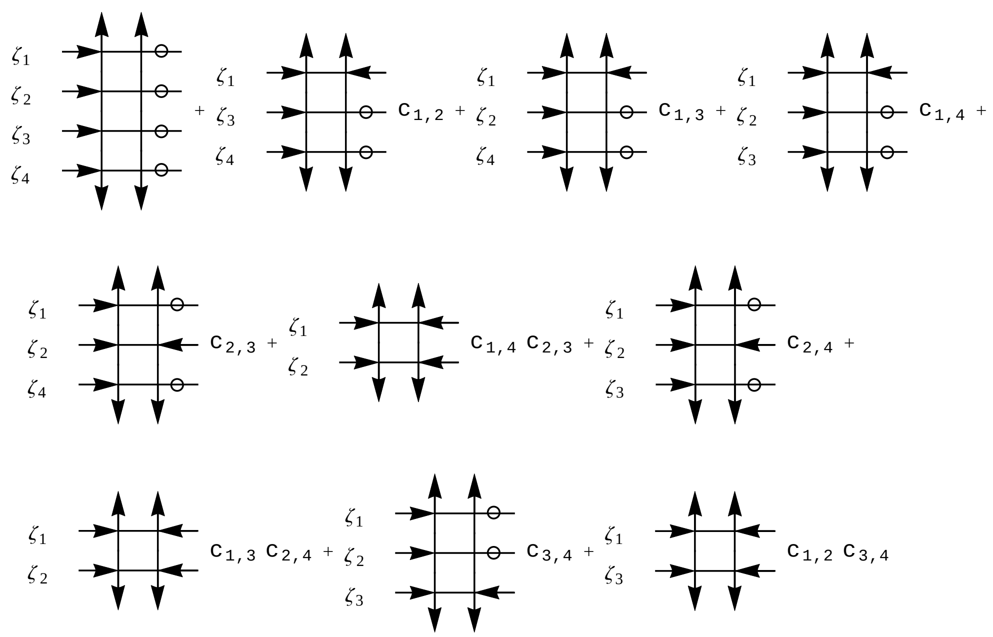


FIGURE 2.7 – This is the highest spin eigenvector of the IK transfer matrix if ζ_i's are the Bethe roots. The coefficients $c_{i,j}$ are functions of the weights in Eq. (2.2.1) and are given in (1.2.7). The circles appearing on the right boundaries signify that the corresponding edges are in the empty state.

the root of unity $q^3 = -1$ we know few terms here, i.e. those corresponding to the domain wall boundaries. The other terms should not be expected to have a nice closed form since they, in general, are not symmetric in ζ's nor in z's. The eigenvector as a whole is symmetric in ζ's and z's. One then needs to find a recurrence relation for it and then, if lucky, it will be possible to find its closed form solution at $q^3 = -1$. The knowledge of this will be helpful in understanding of the other eigenvectors. In particular, we could look at the zero spin eigenvectors at $q^3 = -1$ (i.e. when lower boundary has equal number of up arrows and down arrows). One such eigenvector was computed in the loop basis [57] by means of the quantum Knizhnik–Zamolodchikov equations. In Chapter 5 we consider this vector in the spin basis. We conjecture that this vector is the ground state.

Finally, regarding the generic q expression for Z_N partition function one could try to look for its expansion in terms of symmetric polynomials. For example, it is known that Z^{6v} expands naturally in the Hall–Littlewood polynomials [121, 11, 12].

Chapter 3

Irreducible representations of $U_q(A_2^{(2)})$

This chapter is devoted to the study of certain aspects of the representation theory of the twisted quantum affine algebra $U_q(A_2^{(2)})$. As we will see the finite dimensional representations lead to the solutions of the Yang–Baxter equation. When restricted to the fundamental (three dimensional) representation this gives the IK R-matrix. It is, however, important to understand the higher dimensional representations, as their infinite dimensional limit must be coupled [5, 62] to the problem of diagonalizing the transfer matrix of the IK model. Also, they allow us to build the transfer matrices with higher spin auxiliary spaces, which are related to each other by a series of equations, called the T-systems [86, 88].

The quantum affine algebras (of twisted and untwisted types) is one of the most important classes of the quantum groups. The study of the untwisted algebras can be reduced to a certain extent to the study of the simplest case of $U_q(A_1^{(1)}) = U_q(s\hat{l}_2)$ since the untwisted algebras can be considered as consisting of a number of copies of $U_q(s\hat{l}_2)$ (see [9], Proposition 3.8). To study the twisted algebras one needs to understand also the representation theory of the algebra $U_q(A_2^{(2)})$ [27]. Therefore, there is a strong representation theoretic motivation for studying the finite dimensional representations of the quantum group $U_q(A_2^{(2)})$.

In this chapter we construct the finite dimensional Kirillov–Reshetikhin modules of the algebra $U_q(A_2^{(2)})$ [1]. We will use these results in Chapter 4 to write the R-matrices following the Khoroshkin–Tolstoy approach.

3.1 Definition of the algebra $U_q(A_2^{(2)})$

Let us briefly describe the algebra $A_2^{(2)}$, its universal enveloping $U(A_2^{(2)})$ and the q-deformation $U_q(A_2^{(2)})$. For more details see [71, 21, 26]. To the generalized Cartan

1. We learned recently that these modules were given without a derivation by [43] in a different presentation from the one we use here.

matrix $C = (C_{i,j})_{i,j\in I}$, $I = \{0,1\}$

$$C = \begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix}.$$

we assign the twisted affine Lie algebra $\mathfrak{g}'(C) = A_2^{(2)}$, which is generated by h_i and $e_i^{\pm}$ $(i \in I)$. The defining relations of the algebra $\mathfrak{g}'(C)$ are

$$\begin{gathered} [h_i,h_j] = 0, \\ [h_i,e_j] = C_{i,j}e_{i,j}, \quad [h_i,f_j] = -C_{i,j}f_{i,j}, \\ [e_i,f_j] = \delta_{i,j}h_i, \end{gathered}$$

for all $i,j \in I$, and the Serre relations read

$$(\mathrm{ad}\ e_i)^{1-C_{i,j}}e_j = 0, \quad (\mathrm{ad}\ f_i)^{1-C_{i,j}}f_j = 0,$$

for $i \neq j$, $i,j \in I$. The Cartan subalgebra $\mathfrak{h}'(C)$ of $\mathfrak{g}'(C)$ is generated by the elements h_i. From the above commutation relations between h_i and $e_i^{\pm}$ we can define the weight spaces $\mathfrak{g}'(C)_\gamma$ as follows

$$\mathfrak{g}'(C)_\gamma = \{x \in \mathfrak{g}'(C) \mid [h,x] = \gamma(h)x, \forall\ h \in \mathfrak{h}'(C)\},$$

where γ are the roots which belong to the space $\mathfrak{h}'^*(C)$, dual to the Cartan subalgebra $\mathfrak{h}'(C)$, and the corresponding elements x are the root vectors. The set of roots is

$$\Delta(C) = \{\gamma \in \mathfrak{h}'^*(C) \mid \gamma \neq 0,\, \mathfrak{g}'(C)_\gamma \neq \{0\}\}.$$

Therefore we have the following decomposition of the vector space $\mathfrak{g}'(C)$

$$\mathfrak{g}'(C) = \mathfrak{h}'(C) \oplus \bigoplus_{\gamma\in\Delta(C)} \mathfrak{g}'(C)_\gamma.$$

The generators e_i^+ are root vectors corresponding to the roots denoted by α_i. These roots are called the simple roots and any other root is a linear combination of the simple roots with integer coefficients of the same sign. Thus we have the set of positive $\Delta^+(C) = \sum_{i\in I}\mathbb{N}.\alpha_i$ and the set of negative roots $\Delta^-(C) = -\Delta^+(C)$, so $\Delta(C) = \Delta^-(C) \sqcup \Delta^+(C)$.

The universal enveloping algebra $U(\mathfrak{g}'(C)) = U(A_2^{(2)})$ is the unital associative algebra with generators h_i and $e_i^{\pm}$. These generators satisfy the same defining relations as for $\mathfrak{g}'(C)$ with the Serre relations which can be rewritten as

$$\sum_{r=0}^{1-C_{i,j}} (-1)^r \begin{bmatrix} 1-C_{i,j} \\ r \end{bmatrix} (e_i^{\pm})^r e_j^{\pm} (e_i^{\pm})^{(1-C_{i,j}-r)} = 0.$$

Let q be a generic complex number. In the Drinfeld–Jimbo presentation the quantum deformation of $U(\mathfrak{g}'(C))$ is the algebra $U_q(\mathfrak{g}'(C)) = U_q(A_2^{(2)})$ generated by $e_0^{\pm}$, $e_1^{\pm}$

and k_0, k_1 with the following defining relations

$$k_i k_i^{-1} = k_i^{-1} k_i = 1, \quad k_i k_j = k_j k_i, \tag{3.1.1}$$

$$k_i e_j^{\pm} k_i^{-1} = q^{\pm C_{i,j}} e_j^{\pm}, \tag{3.1.2}$$

$$[e_i^+, e_j^-] = \delta_{i,j} \frac{k_i - k_i^{-1}}{q_i - q_i^{-1}}, \tag{3.1.3}$$

$$\sum_{r=0}^{1-C_{i,j}} (-1)^r \begin{bmatrix} 1 - C_{i,j} \\ r \end{bmatrix}_{q_i} (e_i^{\pm})^r e_j^{\pm} (e_i^{\pm})^{(1-C_{i,j}-r)} = 0, \quad (i \neq j), \tag{3.1.4}$$

where we used the notations

$$[m]_q = \frac{q^m - q^{-m}}{q - q^{-1}}, \qquad [m]_q! = \prod_{j=1}^{m} [j], \qquad \begin{bmatrix} s \\ r \end{bmatrix} = \frac{[s]_q!}{[r]_q! [s-r]_q!},$$

and $q_i = q^{d_i}$ with the numbers $d_0 = 2$, $d_1 = 1/2$ defining the symmetrized Cartan matrix $C^s_{i,j} = C_{i,j} d_i$ $(i = 0, 1)$

$$C^s = \begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix}.$$

The relation between the Cartan elements k_i and the Cartan elements of $U(\mathfrak{g}'(C))$ is given by $k_i = q^{h_i}$. The central element of the algebra $c = k_0 k_1^2$ is set to one, hence from the start we restrict ourselves with the representations of type 1. The algebra $U_q(A_2^{(2)})$ is a Hopf algebra with the comultiplication $\Delta : U_q(A_2^{(2)}) \to U_q(A_2^{(2)}) \otimes U_q(A_2^{(2)})$ given on generators

$$\begin{aligned} \Delta(k_i) &= k_i \otimes k_i, \\ \Delta(e_i^+) &= e_i^+ \otimes k_i + 1 \otimes e_i^+, \\ \Delta(e_i^-) &= e_i^- \otimes 1 + k_i^{-1} \otimes e_i^-. \end{aligned}$$

For the computational purposes a more convenient description of the algebra $U_q(A_2^{(2)})$ is achieved via the Drinfeld presentation [39]. It is given by the generators $x_r^{\pm}$, $h_{\pm m}$ and K $(r \in \mathbb{Z}, m \in \mathbb{Z}_+)$ which satisfy the relations

$$KK^{-1} = K^{-1}K = 1, \quad Kh_k = h_k K, \quad h_k h_l = h_l h_k,$$

$$Kx_k^{\pm} K^{-1} = q^{\pm 1} x_k^{\pm}, \tag{3.1.5}$$

$$[x_r^+, x_s^-] = \frac{\psi_{r+s}^+ - \psi_{r+s}^-}{q - q^{-1}}, \tag{3.1.6}$$

$$[h_r, x_s^{\pm}] = \pm \frac{[r]}{r} (q^r + q^{-r} + (-1)^{r+1}) x_{r+s}^{\pm}, \tag{3.1.7}$$

$$\begin{aligned} &x_{r+2}^{\pm} x_s^{\pm} + (q^{\mp 1} - q^{\pm 2}) x_{r+1}^{\pm} x_{s+1}^{\pm} - q^{\pm 1} x_r^{\pm} x_{s+2}^{\pm} \\ &= q^{\pm 1} x_s^{\pm} x_{r+2}^{\pm} + (q^{\pm 2} - q^{\mp 1}) x_{s+1}^{\pm} x_{r+1}^{\pm} - q^{\pm 1} x_{s+2}^{\pm} x_r^{\pm}, \end{aligned} \tag{3.1.8}$$

$$\mathrm{Sym}(q^{3/2}x^{\pm}_{r\mp1}x^{\pm}_{s}x^{\pm}_{t} - (q^{1/2}+q^{-1/2})x^{\pm}_{r}x^{\pm}_{s\mp1}x^{\pm}_{t} + q^{-3/2}x^{\pm}_{r}x^{\pm}_{s}x^{\pm}_{t\mp1}) = 0, \tag{3.1.9}$$
$$\mathrm{Sym}(q^{-3/2}x^{\pm}_{r\pm1}x^{\pm}_{s}x^{\pm}_{t} - (q^{1/2}+q^{-1/2})x^{\pm}_{r}x^{\pm}_{s\pm1}x^{\pm}_{t} + q^{3/2}x^{\pm}_{r}x^{\pm}_{s}x^{\pm}_{t\pm1}) = 0, \tag{3.1.10}$$

where Sym means a sum over all permutations of r,s and t. The elements $\psi^{\pm}_{k}$ appearing in Eq. (3.1.6) can be written in terms of the generators h_l using the relation

$$\Psi^{\pm}(u) = \sum_{k=0}^{\infty}\psi^{\pm}_{\pm k}u^{\pm k} = K^{\pm1}\exp\Big(\pm(q-q^{-1})\sum_{l=1}^{\infty}h_{\pm l}u^{l}\Big). \tag{3.1.11}$$

We have to stress that often alternative conventions are used. We will adopt these conventions in Chapter 4. The definition we are using here avoids dealing with square roots of q, in order to get the other convention one needs to replace above

$$x^{\pm}_{m} \to \bar{x}^{\pm}_{m}/\sqrt{q^{1/2}+q^{-1/2}}, \quad h_m \to \bar{h}_m/(q^{1/2}+q^{-1/2}), \tag{3.1.12}$$

and after changing $q \to q^2$ we recover the conventions of [27].

The relation between the two presentations is the following

$$e^{+}_{0} = K^{-2}(\bar{x}^{-}_{0}\bar{x}^{-}_{1} - q\bar{x}^{-}_{1}\bar{x}^{-}_{0}), \quad e^{-}_{0} = \frac{1}{[4]^{2}_{q^{1/2}}}(\bar{x}^{+}_{-1}\bar{x}^{+}_{0} - q^{-1}\bar{x}^{+}_{0}\bar{x}^{+}_{-1})K^{2}, \tag{3.1.13}$$
$$e^{+}_{1} = \bar{x}^{+}_{0}, \quad e^{-}_{1} = \bar{x}^{-}_{0}, \quad k_0 = K^{-2}, \quad k_1 = K. \tag{3.1.14}$$

The algebra $U_q(A_2^{(2)})$ has two Borel subalgebras $U_q(\mathfrak{b}^+)$ and $U_q(\mathfrak{b}^-)$, generated by k_i^{+1}, e_i^+ and k_i^{-1}, e_i^-, $i = 0$,1 respectively. Alternatively, using the Drinfeld presentation there are two subalgebras $U_q(\mathfrak{n}^+)$ and $U_q(\mathfrak{n}^-)$, generated by x^+_r and x^-_r, $r \in \mathbb{Z}$ respectively.

3.2 Transfer matrix and q-characters

The purpose of this section is to introduce the q-characters. The q-characters are very important to study the algebraic properties of the transfer matrices and are technically useful in the description of higher dimensional representations $V^{(k)}$ of $U_q(A_2^{(2)})$. We will use the notion of the universal $\mathcal{R}$-matrix and the Khoroshkin–Tolstoy formula. For now this is only needed for the definition of the q-characters as they will be used later in this chapter for the construction of the representations on $V^{(k)}$. In Chapter 4 we will come back to the universal $\mathcal{R}$-matrix and the Khoroshkin–Tolstoy formula.

The universal $\mathcal{R}$-matrix is an element of the tensor product of two copies of a quantum group $\mathcal{A}$. In fact, $\mathcal{R} \in U_q(\mathfrak{b}^+)\otimes U_q(\mathfrak{b}^-)$ for the affine quantum Lie algebras. The universal $\mathcal{R}$-matrix must obey by definition

$$\Delta'(x) = \mathcal{R}\Delta(x)\mathcal{R}^{-1}, \quad \forall x \in \mathcal{A}, \tag{3.2.1}$$
$$(\Delta\otimes\mathrm{id})\mathcal{R} = \mathcal{R}_{1,3}\mathcal{R}_{2,3}, \quad (\mathrm{id}\otimes\Delta)\mathcal{R} = \mathcal{R}_{1,3}\mathcal{R}_{1,2}. \tag{3.2.2}$$

In the first line $\Delta' = \Delta P$, in the second line we have objects in $\mathcal{A}\otimes\mathcal{A}\otimes\mathcal{A}$, then to identify in which two of the three copies of $\mathcal{A}$ the universal $\mathcal{R}$-matrix lives we equip it with indices $\mathcal{R}_{i,j}$, $1 \le i < j \le 3$. In the Hopf algebra Δ' is another coproduct along with Δ, so Eq. (3.2.1) tells us that these two coproducts are related by the universal $\mathcal{R}$ matrix. The conditions (3.2.2) imply that $\mathcal{R}$ satisfies the Yang–Baxter equation (1.1.1). If we specify the auxiliary space to be some representation space $V = V(z)$ and the map $\pi_{V(z)}$ to be the representation of the algebra on the space V, then the monodromy matrix is

$$\mathcal{M}_V(z) = (\pi_{V(z)} \otimes \mathrm{id})\mathcal{R}. \tag{3.2.3}$$

Tracing over the space V gives the transfer matrix

$$T_V(z) = Tr_V \kappa \mathcal{M}_V(z). \tag{3.2.4}$$

Where $T_{V(z)} = T_V(z)$ as well as $M_{V(z)} = M_V(z)$ and κ is a twist which is an element of the Cartan subalgebra. Thus the transfer matrix associates a representation V to $T_V(z) \in U_q(\mathfrak{b}^-)[[z]]$. According to Khoroshkin–Tolstoy [117] and also [91] and [32] the $\mathcal{R}$-matrix can be written as a product of four parts

$$\mathcal{R} = \mathcal{R}^+\mathcal{R}^0\mathcal{R}^-\mathcal{K}. \tag{3.2.5}$$

where $\mathcal{R}^\pm \in U_q(\mathfrak{n}^\pm) \otimes U_q(\mathfrak{n}^\mp)$, $\mathcal{K}$ is constructed from the Cartan elements, and $\mathcal{R}^0$

$$\mathcal{R}^0 = \exp\left(\sum_{m>0} m\ c_m h_m \otimes h_{-m}\right).$$

with some complex numbers c_m. We will come back to Khoroshkin–Tolstoy formula in Section 4.3 where we will write every multiplier explicitly in terms of the generators of the algebra $U_q(A_2^{(2)})$ and proceed further applying this formula to various representations. In order to be able to do so we will need first to construct the representations and the Cartan–Weyl basis of $U_q(A_2^{(2)})$. This we will do in the later sections and in Chapter 4. For the present purpose we will be focused on the part $\mathcal{R}^0$. There exist a homomorphism [53] that relates $T_V(z)$ with the q-character χ_q

$$\chi_q(V(z)) = Tr_V\left((\pi_{V(z)} \otimes 1)\kappa\mathcal{R}^0\mathcal{K}\right). \tag{3.2.6}$$

This homomorphism allows us to relate certain algebraic properties of the transfer matrix and the q-character χ_q, see [53], [60, 61]. As we will learn further a typical matrix element of h_r is of the form

$$\pi_V(h_r) \propto \frac{1}{m}\left(\sum_i a_i^m - \sum_j b_j^m\right). \tag{3.2.7}$$

Therefore, if we denote

$$Y_a = \kappa \exp\left(\sum_{m>0} z^m c_m a^m h_{-m}\right) \tag{3.2.8}$$

and take into account the form (3.2.7), the χ_q becomes a polynomial in $Y_a^{\pm 1}$ with $a \in \mathbb{C}^*$. The q-character for the algebra $U_q(A_2^{(2)})$ is obtained using a certain map between $U_q(A_2^{(1)})$ and $U_q(A_2^{(2)})$ (see Chapter 8 in [61]). The q-character will be used in Section 3.3 to obtain important information about the Kirillov–Reshetikhin modules.

3.3 Kirillov–Reshetikhin modules

Let us briefly recall the classification of the finite dimensional representations of the quantum affine Lie algebras $U_q(\hat{\mathfrak{g}})$ (for more details see [24, 25, 26]). Consider a rank n algebra $U_q(\hat{\mathfrak{g}})$ with the generalized Cartan matrix $C = (C_{i,j})_{i=0,..,n}$ and put $I = \{1,..,n\}$. This algebra [2] in the Drinfeld presentation (see Theorem 2.2 in [25]) is generated by $x^{\pm}_{i,r}$ $(i \in I, r \in \mathbb{Z})$, $k^{\pm}_i$ $(i \in I)$, $h_{i,r}$ $(i \in I, r \in \mathbb{Z} \setminus \{0\})$. These generators obey a similar set of defining relations as the generators of the algebra $U_q(A_2^{(2)})$

$$
\begin{aligned}
&k_i k_i^{-1} = k_i^{-1} k_i = 1, \quad k_i h_j = h_j k_i, \quad h_j h_l = h_l h_j, \\
&k_i x_j^{\pm} k_i = q^{\pm C_{i,j}} x_j^{\pm}, \\
&[x^+_{i,r}, x^-_{j,s}] = \delta_{i,j} \frac{\psi^+_{i,r+s} - \psi^-_{i,r+s}}{q_i - q_i^{-1}}, \\
&[h_{i,r}, x^{\pm}_{j,s}] = \pm \frac{[rC_{i,j}]_{q_i}}{r} x^{\pm}_{j,r+s},
\end{aligned}
$$

$$
x^{\pm}_{i,r+1} x^{\pm}_{j,s} - q_i^{\pm C_{i,j}} x^{\pm}_{j,s} x^{\pm}_{i,r+1} = q_i^{\pm C_{i,j}} x^{\pm}_{i,r} x^{\pm}_{j,s+1} - x^{\pm}_{i,s+1} x^{\pm}_{j,r}
$$

$$
\sum_{\pi \in S_{1-C_{i,j}}} \sum_{k=0}^{1-C_{i,j}} (-1)^k \begin{bmatrix} 1 - C_{i,j} \\ k \end{bmatrix}_{q_i} x^{\pm}_{i,r_{\pi(1)}} \dots x^{\pm}_{i,r_{\pi(k)}} x^{\pm}_{j,s} x^{\pm}_{i,r_{\pi(k+1)}} \dots x^{\pm}_{i,r_{\pi(1-C_{i,j})}} = 0,
$$

if $i \neq j$, S_m is the symmetric group on m letters, $r_1, \dots, r_{1-C_{i,j}}$ are any integers, and $\psi_{i,k}$ are related to $h_{i,l}$ through

$$
\Psi_i^{\pm}(u) = \sum_{k=0}^{\infty} \psi^{\pm}_{i,\pm k} u^{\pm k} = k_i^{\pm 1} \exp\left(\pm (q_i - q_i^{-1}) \sum_{l=1}^{\infty} h_{i,\pm l} u^{\pm l} \right).
$$

Let V be a representation of $U_q(\hat{\mathfrak{g}})$. A vector $v \in V$ is called a highest weight vector in the sense of Drinfeld presentation if it satisfies

$$
x^+_{i,k} v = 0, \qquad \psi^{\pm}_{i,k} v = \psi^{\pm}_{i,k;0} v,
$$

for $i \in I$, $k \in \mathbb{Z}$ and some complex numbers $\psi^{\pm}_{i,k;0}$. This representation is a highest weight representation if, for some highest weight vector v, the whole space V can be generated by the elements of the algebra $V = U_q(\hat{\mathfrak{g}}).v$. The $\{I \times \mathbb{Z}\}$-tuple of numbers $\psi^{\pm}_{i,k;0}$ is called the highest weight of the representation V. The classification of finite dimensional irreducible representations is given by the following theorem

THEOREM (Theorem 3.3 from [25]). *Let $\mathbf{\Psi} = (\psi^{\pm}_{i,k})_{i \in I, k \in \mathbb{Z}}$, then the irreducible representation $V(\mathbf{\Psi})$ is finite dimensional iff there exists $\mathbf{P} = (\mathcal{P}_i)_{i \in I}$ such that*

$$
\sum_{r=0}^{\infty} \psi^+_{i,r} u^r = q^{deg(\mathcal{P}_i)} \frac{\mathcal{P}_i(q_i^{-2} u)}{\mathcal{P}_i(u)} = \sum_{r=0}^{\infty} \psi^-_{i,r} u^{-r},
$$

2. In fact, we consider a quotient of the algebra $U_q(\hat{\mathfrak{g}})$ by the ideal generated by $c - 1$, where $c \in U_q(\hat{\mathfrak{g}})$ is a central element. We loosely call this quotient $U_q(\hat{\mathfrak{g}})$.

in the sense that the left and the right hand terms are the Laurent expansions of the middle term around 0 *and* ∞*, respectively.*

Applying $\mathcal{P}_i(u)$ to a highest weight vector v we get

$$\mathcal{P}_i(u)v = P_i(u)v.$$

The polynomials $\mathcal{P}_i(u)$ must have constant term equal to one. These polynomials are called the Drinfeld polynomials, and the correspondence between V and n-tuples of polynomials $P_1(z)$,..,$P_n(z)$ is bijective [24, 25, 26]. In the case of the twisted quantum affine Lie algebras analogous classification was established in [27].

For the study of integrable models relevant modules are the Kirillov–Reshetikhin modules. These modules are defined through the Drinfeld polynomial of the form

$$P_i(z) = \prod_{l=1}^{k}(1 - aq_i^{k-2l+1}z), \quad P_j(z) = 1 \ (j \neq i), \tag{3.3.1}$$

for some $i = 1,..,n$, where n is the rank of the algebra, $k \in \mathbb{Z}$ and $a \in \mathbb{C}^*$. The parameter a allows us to rescale the variable z, this will be important later when we will consider the limit of our modules $k \to \infty$. Let us get back to the algebra $U_q(A_2^{(2)})$. If $v \in V$ is the highest weight vector, it is annihilated by all x_r^+ operators ($r \in \mathbb{Z}$) and the elements $\psi_k^{\pm}$ act as

$$x_k^+ v = 0, \qquad \psi_k^{\pm} v = \psi_{k;0}^{\pm} v, \tag{3.3.2}$$

for some complex numbers $\psi_{k;0}^{\pm}$. Moreover, all vectors of the space V can be obtained by the action of the algebra $U_q(A_2^{(2)})$ on the highest weight vector $V = U_q(A_2^{(2)}).v$. Such representations V are defined through the Drinfeld polynomial $P(u)$

$$\sum_{k=0}^{\infty} \psi_{\pm k;0}^{\pm} u^{\pm k} = q^{\deg(P)} \frac{P(q^{-1}u)}{P(qu)} = \Psi_0(u). \tag{3.3.3}$$

$\Psi_0(u)$ we call the Drinfeld rational function, so equivalently, we can say that the representations are defined by the Drinfeld rational function. This notion will be important for the asymptotic algebras [62] as we will see later. The first nontrivial representation has the dimension equal to 3. It is called the fundamental representation and it was constructed in [27]. The next higher dimensional representation has the dimension equal to 6 and was constructed in [61]. In general ([61], proposition 10.1), the dimensionality of the representation is equal to the number of tableaux $T \in \mathrm{Tab}_k$ [3] where the set Tab_k, in the case of $U_q(A_2^{(2)})$, consists of tableaux $(T_j)_{1\leq j \leq k}$ with coefficients in $\{0,1,2\}$ satisfying the condition $T_j \leq T_{j+1}$ for $1 \leq j \leq k$. For $k = 1$ each tableaux has only one entry T_1, hence there are three different tableaux $\mathrm{Tab}_1 = \{0,1,2\}$. When $k = 2$, T has two components. If we organize them in columns, the set Tab_2 consists of six columns

$$\mathrm{Tab}_2 = \begin{pmatrix} 0 & 0 & 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 2 & 2 & 2 \end{pmatrix}. \tag{3.3.4}$$

3. In our case the tableaux is simply a sequence of integers. This happens since our algebra is of rank one [61].

It is not difficult to see that the set of tableaux Tab_k correspond to the set of numbers $\{i,j\}$ satisfying the condition $0 \leq i \leq j \leq k$. Indeed, the columns are organized in nondecreasing sequences of 0, 1, 2 and in order to characterise each column we need to know where is the border between zero's and one's and the border between one's and two's, for which the two integers $i \leq j$ is sufficient. The dimensionality of the representation V must be then $(k+1)(k+2)/2$. It is convenient to write the basis of the vector space as $V = \bigoplus_{0\leq n_1 \leq n_2 \leq k} v_{n_1,n_2}$ [4]. Thus, each representation is labeled by the integer k, so $V = V^{(k)}$. We reserve the letter k for this purpose and do not write it explicitly.

The Cartan element K defines the decomposition of V into the weight spaces

$$V = \bigoplus_{p=0}^{2k} V_p, \qquad V_p = \{v_{n_1,n_2} \in V | K v_{n_1,n_2} = q^{k-p} v_{n_1,n_2},\, p = n_1 + n_2\}. \tag{3.3.5}$$

The module V splits into $2k+1$ pieces V_p under the action of K. Each V_p can be written

$$\begin{aligned} V_p &= \{v_{0,p}, v_{1,p-1}, .., v_{\lfloor p/2 \rfloor, \lceil p/2 \rceil}\}, \quad \text{for} \quad p \leq k, \\ V_p &= \{v_{p-k,k}, v_{p-k+1,k-1}, .., v_{\lfloor p/2 \rfloor, \lceil p/2 \rceil}\}, \quad \text{for} \quad p > k. \end{aligned}$$

From Eq. (3.1.5) we see that

$$x_r^- V_p \in V_{p+1}, \quad x_r^+ V_p \in V_{p-1}.$$

If $\mathbf{n} = \{n_1, n_2\}$ and $\mathbf{m}^\pm = \{m_1^\pm, m_2^\pm\}$, such that $n_1 + n_2 \pm 1 = m_1^\pm + m_2^\pm$ and $m_2^- \leq k$, $m_1^+ \geq 0$, then we can write the action of the elements $x_r^\pm$ as

$$x_r^\pm v_{\mathbf{n}} = \sum_{\mathbf{m}^\pm} x_r^\pm{}_{\mathbf{n},\mathbf{m}^\pm} v_{\mathbf{m}^\pm}, \tag{3.3.6}$$

where $x_r^\pm{}_{\mathbf{n},\mathbf{m}}$ are the corresponding matrix elements of $x_r^\pm$. The space V is best seen as a graph, where nodes are the vectors v_{n_1,n_2} and the edges can be viewed as the matrix elements of $x_r^\pm$, thus the edges show which vectors are related by the action of a single operator $x^\pm$ (see Fig. 3.1). In order to make use of the matrix representations we need a single-indexed labelling of the basis elements of $V = \{u_0, .., u_L\}$, where $L = (k+1)(k+2)/2 - 1$. We also need an ordering on vectors v_{n_1,n_2}. We choose the following ordering

$$\begin{aligned} v_{n_1,n_2} &\prec v_{m_1,m_2}, \qquad \text{if } m_2 > n_2, \\ v_{n_1,n} &\prec v_{m_1,n}, \qquad \text{if } m_1 > n_1. \end{aligned} \tag{3.3.7}$$

The basis vectors u_i have all components equal to zero except from the one at position i which is equal to 1. The map from u_i to v_{n_1,n_2} is a little bit complicated, we need first to introduce some functions

$$\gamma_s = \lfloor \sqrt{4s+1} \rfloor, \tag{3.3.8}$$

$$\epsilon_s = \lfloor \frac{1}{4} \lceil \sqrt{4s+4} \rceil^2 \rfloor - s, \tag{3.3.9}$$

$$\delta_s = s + \frac{1 - (-1)^{\gamma_s} - 2\gamma_s^2}{8}. \tag{3.3.10}$$

4. An alternative basis can be used, see [43].

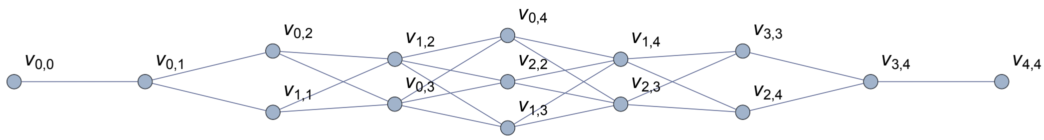


FIGURE 3.1 – The graph of the vector space V for $k = 4$. Vertices represent basis vectors v_{n_1,n_2}, edges connect those vectors which can be obtained one from another by the action of $x_r^\pm$. Vectors belonging to the same subspace V_p are aligned vertically.

With these definitions we have $u_i \to v_{n_1(i),n_2(i)}$, where

$$(n_1(i),n_2(i)) = (\delta_i,\gamma_i - \delta_i - 1), \quad \text{if } i \leq k, \tag{3.3.11}$$
$$(n_1(i),n_2(i)) = (k - \gamma_{L-i-1} + \epsilon_{L-i-1},k + 1 - \epsilon_{L-i-1}), \quad \text{if } i > k. \tag{3.3.12}$$

Here is an example of this map

$$u_0 \to v_{0,0},\ u_1 \to v_{0,1},\ u_2 \to v_{0,2},\ u_3 \to v_{1,1},\ u_4 \to v_{0,3},$$
$$u_5 \to v_{1,2},\ u_6 \to v_{1,3},\ u_7 \to v_{2,2},\ u_8 \to v_{2,3},\ u_9 \to v_{3,3}.$$

With this notation the action of $x_r^\pm$ becomes

$$x_r^\pm u_i = \sum_j {x_r^\pm}_{i,j} u_j. \tag{3.3.13}$$

This action can be summarized in the directed graphs. For $k = 3$ see Fig. 3.2 for the action of x_r^-. The graph of x_r^+ is the same but with the arrows reversed. The adjacency

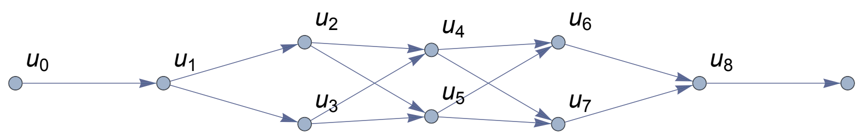


FIGURE 3.2 – The directed graph representing the action of x_r^- on $V = \{u_0,..,u_9\}$.

matrix of these directed graphs give us the matrix representation of the elements $x_r^\pm$.

The matrix corresponding to the graph in Fig. 3.2 is

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ x^-_{0,1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & x^-_{1,2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & x^-_{1,3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & x^-_{2,4} & x^-_{3,4} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & x^-_{2,5} & x^-_{3,5} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & x^-_{4,6} & x^-_{5,6} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & x^-_{4,7} & x^-_{5,7} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & x^-_{6,8} & x^-_{7,8} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x^-_{8,9} & 0 \end{pmatrix}.$$

Our goal is to obtain a formula for all $x^{\pm}_{r\ i,j}$ for any k for the Kirillov–Reshetikhin modules. Before doing this we need to understand the action of the elements $\psi_k^{\pm}$ on V.

Let us write the second equation in (3.3.2) for any v_{n_1,n_2} assuming that V is irreducible

$$\psi_k^{\pm} v_{n_1,n_2} = \psi^{\pm}_{k\ n_1,n_2} v_{n_1,n_2}. \tag{3.3.14}$$

Then we can write

$$\sum_{k=0}^{\infty} u^{\pm k} \psi^{\pm}_{k\ n_1,n_2} = q^{\deg(P_{n_1,n_2})-\deg(Q_{n_1,n_2})} \frac{P_{n_1,n_2}(q^{-1}u)}{P_{n_1,n_2}(qu)} \frac{Q_{n_1,n_2}(qu)}{Q_{n_1,n_2}(q^{-1}u)} = \Psi^{\pm}_{n_1,n_2}(u). \tag{3.3.15}$$

It can be shown [27] that $\Psi^{+}_{n_1,n_2}(u) = \Psi^{-}_{n_1,n_2}(u)$, so we will write simply $\Psi_{n_1,n_2}(u)$ for both. For $n_1 = n_2 = 0$, i.e. for the highest weight vector, $Q_{0,0}(u) = 1$ and $P_{0,0}(u) = P(u)$ and we also assume $\Psi_{0,0}(u) = \Psi_0(u)$ to match the notation in (3.3.3). We will study the Kirillov–Reshetikhin modules which are defined for $U_q(A_2^{(2)})$ by

$$P(u) = (1 - au)(1 - aq^2u)..(1 - aq^{2k-2}u). \tag{3.3.16}$$

To describe explicitly the $V^{(k)}$ representations of $U_q(A_2^{(2)})$ we first compute all $\Psi^{\pm}_{n_1,n_2}(u)$, or equivalently, the polynomials P_{n_1,n_2} and Q_{n_1,n_2}. Most conveniently this could be done using the knowledge of the q-characters. Using the result of [61], proposition 10.2, we can write explicitly the the q-character for the module $V^{(k)}$ of $U_q(A_2^{(2)})$

$$\chi_q(V) = \sum_{n_2=0}^{k} \sum_{n_1=0}^{n_2} \prod_{j=1}^{k-n_2} Z_{aq^{2(j-1)}} \prod_{j=k-n_2+1}^{k-n_1} \frac{Z_{-aq^{2j-1}}}{Z_{aq^{2j}}} \prod_{j=k-n_1+1}^{k} \frac{1}{Z_{-aq^{2j+1}}}, \tag{3.3.17}$$

where, following the conventions for twisted algebras of [61], we replaced variables Y_a with Z_a with respect to Eq. (3.2.8). Each summand in (3.3.17) is labeled by n_1 and n_2. After the substitution

$$Z_x \to \frac{P(q^{-1}x)}{P(qx)}, \quad \text{where } P(u) = 1 - au, \tag{3.3.18}$$

we obtain the functions $\Psi_{n_1,n_2}(u)$. Indeed, the summation in (3.3.17) occurs because of the trace in (3.2.6), thus each summand represents a matrix element of $\pi_V(h_r)$. Recall the form of $\pi_V(h_r)$ Eq. (3.2.7). Each summand in (3.3.17) is a product of the form $\prod_{i,j} Z_{a_i} Z_{b_j}^{-1}$, where a_i and b_j correspond to the numbers a_i and b_j appearing in Eq. (3.2.7). Thus the q-characters give the matrix elements of the operators h_r and Eq. (3.1.11) allows us to compute the polynomials P_{n_1,n_2} and Q_{n_1,n_2} appearing in (3.3.15)

$$P_{n_1,n_2}(u) = \prod_{j=1}^{k-n_2} (1 - auq^{2(j-1)}) \prod_{j=k-n2+1}^{k-n_1} (1 + auq^{2j-1}), \tag{3.3.19}$$

$$Q_{n_1,n_2}(u) = \prod_{j=k-n_2+1}^{k-n_1} (1 - auq^{2j}) \prod_{j=k-n1+1}^{k} (1 + auq^{2j+1}). \tag{3.3.20}$$

For $n_1 = n_2 = 0$ we recover (3.3.16) and $Q_{0,0}(u) = 1$. Plugging (3.3.19) and (3.3.20) into (3.3.15) we get the formula for $\Psi_{n_1,n_2}(u)$

$$\Psi_{n_1,n_2}(u) = \frac{(1 - auq^{-1}) \left(1 + auq^{2k+2}\right) \left(1 - auq^{2k-2n_1+1}\right) \left(1 + auq^{2k-2n_2}\right)}{(1 + auq^{2k-2n_1}) (1 + auq^{2k-2n_1+2}) (1 - auq^{2k-2n_2-1}) (1 - auq^{2k-2n_2+1})}. \tag{3.3.21}$$

Using this formula we can write the matrix elements of ψ_l^+ (and also ψ_l^-)

$$\begin{aligned}
\psi^+_{l\ n_1,n_2} = {} & \frac{q^{2(n_1+n_2)}}{q-1} \times \\
\Bigg(& \frac{q^{-2(k+n_1)}(-aq^{2k-2n_1+2})^l (q^{2n_1}-1)(q^{2n_1}-q^{2n_2+2})(q^{2k+3}+q^{2n_1})}{(q^{2n_1}+q^{2n_2+1})(q^{2n_1}+q^{2n_2+3})} - \\
& \frac{q^{-2(k+n_1)}(-aq^{2(k-n_1)})^l (q^{2n_1+2}-1)(q^{2n_1}-q^{2n_2})(q^{2k+1}+q^{2n_1})}{(q^{2n_1+1}+q^{2n_2})(q^{2n_1}+q^{2n_2+1})} - \\
& \frac{q^{-2(k+n_2)}(aq^{2k-2n_2-1})^l (q^{2n_2+2}-q^{2n_1})(q^{2n_2+3}+1)(q^{2n_2}-q^{2k})}{(q^{2n_1}+q^{2n_2+1})(q^{2n_1}+q^{2n_2+3})} + \\
& \frac{q^{-2(k+n_2)}(aq^{2k-2n_2+1})^l (q^{2n_2}-q^{2n_1})(q^{2n_2+1}+1)(q^{2n_2}-q^{2k+2})}{(q^{2n_1+1}+q^{2n_2})(q^{2n_1}+q^{2n_2+1})} \Bigg).
\end{aligned}$$

Later on, when we will consider the "bosonic" modes of the Drinfeld generators we will rewrite these matrix elements in a nicer form. The matrix elements of h_r are important for further calculations, they have the form

$$h_{i,j}^{(r)} = \frac{a^r q^{2kr+1} \left(A_r \left((-1)^r q^{r-2ir} + q^{-2jr}\right) - q^{(-2k-1)r} + (-1)^{r+1} q^{2r}\right)}{(q^2-1)r}, \tag{3.3.22}$$

where

$$A_r = q^{-r} + q^r + (-1)^{r+1}. \tag{3.3.23}$$

In the next section we show the following result

THEOREM. *The matrix elements of the operators $x_m^{\pm}$ in the representation space $V^{(k)}$ are given by the formulae*

$$\begin{aligned} x_m^+ v_{i,j} &= A(-1)^m q^{2m-2im} v_{i-1,j} \\ &+ B^{-1} \frac{\left(q^{2j+1}+1\right)\left(q^{2j}-q^{2i}\right)\left(q^{2k+2}-q^{2j}\right) q^{i+(1-2j)m-j-k+1}}{(q-1)^2(q+1)\left(q^{2i+1}+q^{2j}\right)\left(q^{2i}+q^{2j+1}\right)} v_{i,j-1}, \end{aligned} \tag{3.3.24}$$

$$\begin{aligned} x_m^- v_{i,j} &= A^{-1} \frac{(-1)^m \left(q^{2i+2}-1\right)\left(q^{2j}-q^{2i}\right)\left(q^{2i+1}+q^{2k+2}\right) q^{-2im-i+j-k}}{(q-1)^2(q+1)\left(q^{2i+1}+q^{2j}\right)\left(q^{2i}+q^{2j+1}\right)} v_{i+1,j} \\ &+ Bq^{-(2j+1)m} v_{i,j+1}, \end{aligned} \tag{3.3.25}$$

where A and B are free parameters.
In the above formulae we made a choice $a = q^{-2k}$. The parameter a enters as a common factor in both formulae (3.3.24) and (3.3.25), so it can be easily recovered.

3.4 Commutation relations

In order to find the representations of $U_q(A_2^{(2)})$ we need to "solve" Eqs. (3.1.6)-(3.1.10). We start with Eq. (3.1.7) where we put $r = 1$

$$[h_1, x_s^+] = A_1 x_{s+1}^+, \tag{3.4.1}$$
$$[h_1, x_s^-] = -A_1 x_{s+1}^-. \tag{3.4.2}$$

Let us focus on the equation with x^+. Consider the action of x^+ and h_r on the vector v_{n_1,n_2}

$$\begin{aligned} x_s^+ v_{n_1,n_2} &= \sum_i \alpha_{n_1-i,n_2+i}(s) v_{n_1-i-1,n_2+i} + \sum_i \beta_{n_1-i,n_2+i}(s) v_{n_1+i,n_2-i-1}, \\ h_r v_{n_1,n_2} &= h_{n_1,n_2}^{(r)} v_{n_1,n_2}, \end{aligned}$$

where $i \geq 0$ and $\alpha_{i,j}(s)$ and $\beta_{i,j}(s)$ are the matrix elements which vanish whenever the indices of the corresponding vectors $v_{n,m}$ do not satisfy $0 \leq n \leq m \leq k$. Acting with Eq. (3.4.1) on v_{n_1,n_2} we obtain the recurrence relations

$$\begin{aligned} (h_{-i+n_1-1,i+n_2}^{(1)} - h_{n_1,n_2}^{(1)}) \alpha_{n_1-i,i+n_2}(s) - A_1 \alpha_{n_1-i,i+n_2}(s+1) = 0, \\ (h_{i+n_1,-i+n_2-1}^{(1)} - h_{n_1,n_2}^{(1)}) \beta_{i+n_1,n_2-i}(s) - A_1 \beta_{i+n_1,n_2-i}(s+1) = 0. \end{aligned}$$

These relations are easily solved

$$\alpha_{n_1-i,i+n_2}(s) = \alpha_{n_1-i,i+n_2} \left(\frac{h_{-i+n_1-1,i+n_2}^{(1)} - h_{n_1,n_2}^{(1)}}{A_1} \right)^s, \tag{3.4.3}$$

$$\beta_{i+n_1,n_2-i}(s) = \beta_{i+n_1,n_2-i} \left(\frac{h_{i+n_1,-i+n_2-1}^{(1)} - h_{n_1,n_2}^{(1)}}{A_1} \right)^s. \tag{3.4.4}$$

We set here $\alpha_{n,m}(0) = \alpha_{n,m}$ and $\beta_{n,m}(0) = \beta_{n,m}$. Let us take the same equation (3.1.7) for arbitrary r and act on v_{n_1,n_2}. We obtain again two equations

$$\alpha_{n_1-i,i+n_2}(s)\left(h^{(r)}_{-i+n_1-1,i+n_2} - h^{(r)}_{n_1,n_2}\right) - \alpha_{n_1-i,i+n_2}(r+s)\frac{A_r[r]}{r} = 0,$$
$$\beta_{i+n_1,n_2-i}(s)\left(h^{(r)}_{i+n_1,-i+n_2-1} - h^{(r)}_{n_1,n_2}\right) - \beta_{i+n_1,n_2-i}(r+s)\frac{A_r[r]}{r} = 0.$$

Plugging (3.4.3) and (3.4.4) into these equations gives

$$\left(h^{(r)}_{-i+n_1-1,i+n_2} - h^{(r)}_{n_1,n_2}\right) - \left(\frac{h^{(1)}_{-i+n_1-1,i+n_2} - h^{(1)}_{n_1,n_2}}{A_1}\right)^r \frac{A_r[r]}{r} = 0, \tag{3.4.5}$$
$$\left(h^{(r)}_{i+n_1,-i+n_2-1} - h^{(r)}_{n_1,n_2}\right) - \left(\frac{h^{(1)}_{i+n_1,-i+n_2-1} - h^{(1)}_{n_1,n_2}}{A_1}\right)^r \frac{A_r[r]}{r} = 0. \tag{3.4.6}$$

We can simplify this by introducing

$$\xi^{(r)}_i = \frac{h^{(r)}_{-i+n_1-1,i+n_2} - h^{(r)}_{n_1,n_2}}{A_r[r]} r, \tag{3.4.7}$$
$$\chi^{(r)}_i = \frac{h^{(r)}_{i+n_1,-i+n_2-1} - h^{(r)}_{n_1,n_2}}{A_r[r]} r, \tag{3.4.8}$$

which brings (3.4.5) and (3.4.6) to

$$\xi^{(r)}_i = (\xi^{(1)}_i)^r, \quad \chi^{(r)}_i = (\chi^{(1)}_i)^r. \tag{3.4.9}$$

On the other hand, plugging (3.3.22) into (3.4.7) and (3.4.8) we can compute $\xi^{(r)}_i$ and $\chi^{(r)}_i$

$$\xi^{(r)}_i = \frac{a^r q^{2kr+1}\left((-1)^r\left(q^{2(i+1)r} - 1\right)q^{r-2n_1r} + \left(q^{-2ir} - 1\right)q^{-2n_2r}\right)}{\left(q^2-1\right)[r]},$$
$$\chi^{(r)}_i = \frac{a^r q^{2kr+1}\left((-1)^{r+1}\left(q^{2ir} - 1\right)q^{-2ir-2n_1r+r} + \left(q^{2(i+1)r} - 1\right)q^{-2n_2r}\right)}{\left(q^2-1\right)[r]}.$$

Eqs. (3.4.9) are satisfied only when $i = 0$ ($i = -1$ also works, but we agreed that $i \geq 0$), which drastically simplifies the pictures in Fig. 3.1 and Fig. 3.2 (compare with Fig. 3.3). The computations for x^-_r follow the same route. Now we can write the action of the operators $x^\pm_r$ in a more compact form. We equip the coefficients α and β with the super scripts $+$ and $-$ as we need to distinguish the matrix elements of x^+ and x^-. Since the graded operators $x^\pm_r$ can be reduced to $x^\pm = x^\pm_0$ using (3.4.3) and (3.4.4) we will be focused on computing the matrix elements of $x^\pm$. The action of $x^\pm$ becomes

$$x^\pm v_{n_1,n_2} = \alpha^\pm_{n_1,n_2} v_{n_1\mp1,n_2} + \beta^\pm_{n_1,n_2} v_{n_1,n_2\mp1}. \tag{3.4.10}$$

Note that $\beta^-_{i,k} = \alpha^+_{0,i} = 0$ and $\beta^+_{i,i} = \alpha^-_{i,i} = 0$ for all i. The graph representing the vector space V with $k = 6$ is presented in Fig. 3.3. In the rest of this section we will

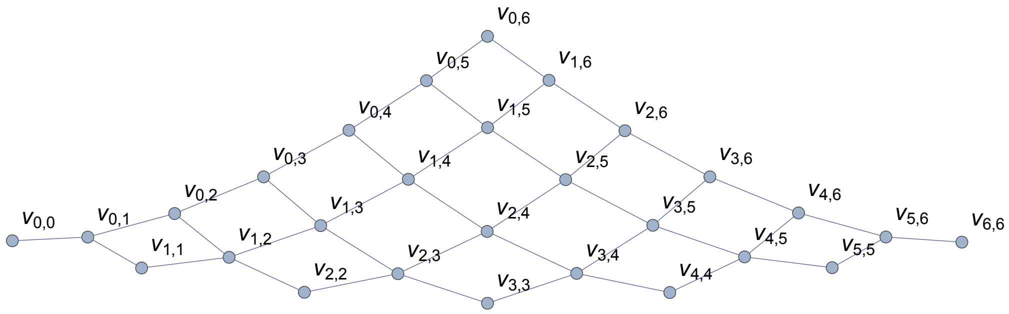


FIGURE 3.3 – The graph of the vector space V for $k = 6$.

focus on solving the relations (3.1.6) for $r = s = 0$ and (3.1.8) again for $r = s = 0$. As it turns out this is enough to determine the matrices $x^{\pm}$.

First, for simplicity we will assume $\beta^-_{n_1,n_2} = 1$, this is possible since we can rescale the basis vectors v_{n_1,n_2}. Indeed, if the new basis is $\tilde{v}_{n_1,n_2} = \delta_{n_1,n_2} v_{n_1,n_2}$ with some coefficients δ_{n_1,n_2}, then acting with $x^{\pm}$ we get

$$\begin{aligned}
x^{\pm}\tilde{v}_{n_1,n_2} &= \tilde{\alpha}^{\pm}_{n_1,n_2}\tilde{v}_{n_1\mp 1,n_2} + \tilde{\beta}^{\pm}_{n_1,n_2}\tilde{v}_{n_1,n_2\mp 1},\\
x^{\pm}v_{n_1,n_2} &= \tilde{\alpha}^{\pm}_{n_1,n_2}\frac{\delta_{n_1\mp 1,n_2}}{\delta_{n_1,n_2}}v_{n_1\mp 1,n_2} + \tilde{\beta}^{\pm}_{n_1,n_2}\frac{\delta_{n_1,n_2\mp 1}}{\delta_{n_1,n_2}}v_{n_1,n_2\mp 1}.
\end{aligned}$$

Comparing the last equation with Eq. (3.4.10) we obtain, in particular,

$$\begin{aligned}
\delta_{n_1,n_2} &= \delta_{n_1,n_2-1}\frac{\tilde{\beta}^+_{n_1,n_2}}{\beta^+_{n_1,n_2}},\\
\delta_{n_1,n_2} &= \delta_{n_1,n_2+1}\frac{\tilde{\beta}^-_{n_1,n_2}}{\beta^-_{n_1,n_2}}.
\end{aligned}$$

Since both equations define the same ratio $\delta_{n_1,n_2}/\delta_{n_1,n_2-1}$ we get the constraint

$$\tilde{\beta}^-_{n_1,n_2-1}\tilde{\beta}^+_{n_1,n_2} = \beta^-_{n_1,n_2-1}\beta^+_{n_1,n_2}.$$

Which means that in the new basis β^- can be set to a constant. Similar argument also holds for the coefficients $\alpha^{\pm}$.

Take Eq. (3.1.6), set there $r = s = 0$ and act on v_{n_1,n_2}, we obtain three equations, two homogeneous

$$\alpha^+_{n_1,n_2+1}\beta^-_{n_1,n_2} - \alpha^+_{n_1,n_2}\beta^-_{n_1-1,n_2} = 0, \tag{3.4.11}$$
$$\alpha^-_{n_1,n_2}\beta^+_{n_1+1,n_2} - \alpha^-_{n_1,n_2-1}\beta^+_{n_1,n_2} = 0, \tag{3.4.12}$$

and one inhomogeneous

$$\begin{aligned}
-\alpha^-_{n_1-1,n_2}\alpha^+_{n_1,n_2} + \alpha^-_{n_1,n_2}\alpha^+_{n_1+1,n_2} - \beta^-_{n_1,n_2-1}\beta^+_{n_1,n_2} + \beta^-_{n_1,n_2}\beta^+_{n_1,n_2+1}&\\
= \frac{K_{n_1,n_2} - K^{-1}_{n_1,n_2}}{q - q^{-1}} = [k - n_1 - n_2].&
\end{aligned} \tag{3.4.13}$$

The two homogeneous equations can be solved easily, we get

$$\alpha^+_{n_1,n_2} = \alpha^+_{n_1,k}, \tag{3.4.14}$$

$$\beta^+_{n_1,n_2} = \beta^+_{n_2-1,n_2} \prod_{s=n_1}^{n_2-2} \frac{\alpha^-_{s,n_2}}{\alpha^-_{s,n_2-1}}. \tag{3.4.15}$$

Before turning to the inhomogeneous equation (3.4.13) we can solve another homogeneous equation coming from (3.1.8) where we set again $r = s = 0$. For higher graded elements $x^\pm_r$ we have

$$x^\pm_r v_{n_1,n_2} = \alpha^\pm_{n_1,n_2}(r) v_{n_1\mp1,n_2} + \beta^\pm_{n_1,n_2}(r) v_{n_1,n_2\mp1}. \tag{3.4.16}$$

The coefficients $\alpha^\pm_{n_1,n_2}(r)$ and $\beta^\pm_{n_1,n_2}(r)$ as we learned from (3.4.3) and (3.4.4) are proportional to $\alpha^\pm_{n_1,n_2}$ and $\beta^\pm_{n_1,n_2}$ respectively

$$\alpha^\pm_{n_1,n_2}(r) = f^\pm_{n_1,n_2}(r)\alpha^\pm_{n_1,n_2}, \quad \beta^\pm_{n_1,n_2}(r) = g^\pm_{n_1,n_2}(r)\beta^\pm_{n_1,n_2}, \tag{3.4.17}$$

where

$$f^+_{n_1,n_2}(r) = (-aq^{2(k-n_1+1)})^r, \qquad f^-_{n_1,n_2}(r) = (-aq^{2(k-n_1)})^r, \tag{3.4.18}$$

$$g^+_{n_1,n_2}(r) = (aq^{2(k-n_2)+1})^r, \qquad g^-_{n_1,n_2}(r) = (aq^{2(k-n_2)-1})^r. \tag{3.4.19}$$

From (3.1.8) for x^- elements we obtain three equations, two of which are trivially satisfied and the remaining one reads

$$\begin{aligned} &\alpha^-_{n_1,n_2+1}\beta^-_{n_1,n_2}\left(\left(q-q^{-2}\right) f^-_{n_1,n_2+1}(1) g^-_{n_1,n_2}(1) + f^-_{n_1,n_2+1}(2) - q^{-1} g^-_{n_1,n_2}(2)\right) + \\ &\alpha^-_{n_1,n_2}\beta^-_{n_1+1,n_2}\left(\left(q-q^{-2}\right) f^-_{n_1,n_2}(1) g^-_{n_1+1,n_2}(1) - q^{-1} f^-_{n_1,n_2}(2) + g^-_{n_1+1,n_2}(2)\right) = 0, \end{aligned} \tag{3.4.20}$$

or in a more compact form

$$\alpha^-_{n_1,n_2+1}\beta^-_{n_1,n_2} + \phi_{n_1,n_2}\alpha^-_{n_1,n_2}\beta^-_{n_1+1,n_2} = 0, \tag{3.4.21}$$

with

$$\phi_{n_1,n_2} = \frac{\left(q^{2n_1+1} + q^{2n_2}\right)\left(q^{2n_1} - q^{2n_2+2}\right)}{\left(q^{2n_1} - q^{2n_2}\right)\left(q^{2n_1} + q^{2n_2+3}\right)}. \tag{3.4.22}$$

Eq. (3.4.21) is solved by

$$\alpha^-_{n_1,n_2} = \alpha^-_{n_1,k} \prod_{s=n_2}^{k-1} \phi^{-1}_{n_1,s}. \tag{3.4.23}$$

We can now rewrite (3.4.15) as

$$\beta^+_{n_1,n_2} = \beta^+_{n_2-1,n_2} \prod_{s=i}^{n_2-2} \phi_{s,n_2-1}. \tag{3.4.24}$$

The remaining unknowns are: $\beta^+_{j-1,j}$, $\alpha^+_{j,k}$ and $\alpha^-_{i,k}$. Eq. (3.4.13) at $n_1 = n_2$ becomes

$$\begin{aligned}&\alpha^-_{n_2,n_2}\alpha^+_{n_2+1,n_2} - \alpha^-_{n_2-1,n_2}\alpha^+_{n_2,n_2}\\ &- \beta^-_{n_2,n_2-1}\beta^+_{n_2,n_2} + \beta^-_{n_2,n_2}\beta^+_{n_2,n_2+1} = [k-2n_2].\end{aligned} \tag{3.4.25}$$

Two terms from the left hand side vanish since $\beta^-_{n_2,n_2-1} = \beta^+_{n_2,n_2} = 0$ and $\alpha^-_{n_2,n_2} = \alpha^+_{n_2+1,n_2} = 0$, also we recall $\beta^-_{n_1,n_2} = 1$ and set

$$\gamma_j = \alpha^-_{j-1,k}\alpha^+_{j,k}, \tag{3.4.26}$$

then, taking into account (3.4.14) and (3.4.23) gives

$$-\alpha^-_{j-1,k}\alpha^+_{j,k}\prod_{s=j}^{k-1}\phi^{-1}_{j-1,s} + \beta^+_{j,j+1} = [k-2j], \tag{3.4.27}$$

so we obtain $\beta^+_{j,j+1}$

$$\beta^+_{j,j+1} = [k-2j] + \gamma_j\prod_{s=j}^{k-1}\phi^{-1}_{j-1,s}. \tag{3.4.28}$$

It remains to compute the numbers γ_j. We take the same equation (3.4.13) where we set $n_2 = n_1+1$

$$\begin{aligned}&-\alpha^-_{n_1-1,n_1+1}\alpha^+_{n_1,n_1+1} + \alpha^-_{n_1,n_1+1}\alpha^+_{n_1+1,n_1+1} -\\ &\beta^-_{n_1,n_1}\beta^+_{n_1,n_1+1} + \beta^-_{n_1,n_1+1}\beta^+_{n_1,n_1+2} = [k-2n_1-1].\end{aligned} \tag{3.4.29}$$

Plug here what we already computed and obtain

$$\begin{aligned}&-\gamma_j\prod_{s=j+1}^{k-1}\phi^{-1}_{j-1,s} + \gamma_{j+1}\prod_{s=j+1}^{k-1}\phi^{-1}_{j,s} + \gamma_{j+1}\prod_{s=j+2}^{k-1}\phi^{-1}_{j,s} - \gamma_j\prod_{s=j}^{k-1}\phi^{-1}_{j-1,s}\\ &= [k-2j-1] + [k-2j] - [k-2j-2]\phi_{j,j+1}\end{aligned} \tag{3.4.30}$$

Now we can solve this recurrence relation for γ_j

$$\begin{aligned}\gamma_j = &\prod_{i=1}^{j-1}\frac{(\phi_{i-1,i}+1)\prod_{s=i+1}^{k-1}\phi_{i,s}}{\phi_{i-1,i}(\phi_{i,i+1}+1)\prod_{s=i+1}^{k-1}\phi_{i-1,s}}\left(\sum_{m=0}^{j-1}\prod_{i=1}^{m}\frac{\phi_{i-1,i}(\phi_{i,i+1}+1)\prod_{s=i+1}^{k-1}\phi_{i-1,s}}{(\phi_{i-1,i}+1)\prod_{s=i+1}^{k-1}\phi_{i,s}}\times\right.\\ &\left.\frac{\prod_{s=m+1}^{k-1}\phi_{m,s}}{\phi_{m,m+1}+1}\left(-[2k-2m-2]\phi_{m,m+1} + [2k-2m] + [2k-2m-1]\right) + \mathrm{const}\right)\end{aligned} \tag{3.4.31}$$

Note that $\phi_{i,j} = \phi_{j-i}$ depends on the difference $j-i$ which can be seen from the formula (3.4.22). Using that, performing some simplifications in (3.4.31) and setting $\mathrm{const} = 0$ since $\gamma_0 = 0$, we get

$$\gamma_j = \frac{\prod_{s=j}^{k-1}\phi_{-j+s+1}\left(\sum_{n=0}^{2j-1}[k-n] - \phi_1\sum_{n=1}^{j}[k-2n]\right)}{\phi_1+1} \tag{3.4.32}$$

The knowledge of γ_j gives us the solution to all commutation relations of the Drinfeld presentation (3.1.6)-(3.1.10). Let us summarize the results of this section. Recall that we put $\beta^-_{n_1,n_2}$ to a constant equal to 1. Now we would like to recover it, set $\beta^-_{n_1,n_2} = B$. We also set $\alpha^+_{i,k} = A$. The matrix elements $\alpha^\pm$ and $\beta^\pm$ become

$$\begin{aligned}
&\alpha^+_{i,j} = A, \qquad \beta^-_{i,j} = B,\\
&\alpha^-_{i,j} = A^{-1}\frac{q^{-i+j+k+2}\left(q^{2(i+1)}-1\right)\left(q^{2j}-q^{2i}\right)\left(q^{2i-2k+1}+q^{-2}\right)}{(q-1)^2(q+1)\left(q^{2i+1}+q^{2j}\right)\left(q^{2i}+q^{2j+1}\right)},\\
&\beta^+_{i,j} = B^{-1}\frac{q^{3+k+i-j}\left(q^{2j+1}+1\right)\left(q^{2j}-q^{2i}\right)\left(q^{-2}-q^{2j-2k}\right)}{(q-1)^2(q+1)\left(q^{2i+1}+q^{2j}\right)\left(q^{2i}+q^{2j+1}\right)}.
\end{aligned}\tag{3.4.33}$$

Putting this together with (3.4.18) and (3.4.19) into (3.4.17) we obtain the action of $x^\pm$ on $V^{(k)}$ for arbitrary k and thus obtain the formulae (3.3.24) and (3.3.25).

3.5 Asymptotic representations

It is known [101, 60] that the characters of the Kirillov–Reshetikhin modules have a well defined limit when $k \to \infty$. This limit allows us to obtain the so called asymptotic representations of the asymptotic algebra $\tilde{U}$ [62]. This algebra differs from the quantum group by the condition that its Cartan element analogous to k is not invertible. With this condition it is possible to avoid the divergences appearing when $k \to \infty$. Indeed, one of the two elements $Kv_{n_1,n_2} \propto q^k v_{n_1,n_2}$, $K^{-1}v_{n_1,n_2} \propto q^{-k}v_{n_1,n_2}$ is always divergent.

Assume $|q| > 1$, the algebra $\tilde{U}$ is given by the generators $\tilde{x}^\pm_r$, $\tilde{h}_{\pm m}$ and κ_0 ($r \in \mathbb{Z}$, $m \in \mathbb{Z}_+$) and relations similar to $U_q(A_2^{(2)})$

$$\tilde{\psi}^+_0 = 1, \qquad \tilde{\psi}^-_0 = \kappa_0^2 \tag{3.5.1}$$

$$\kappa_0\tilde{h}_r = \tilde{h}_r\kappa_0, \quad \tilde{h}_r\tilde{h}_l = \tilde{h}_l\tilde{h}_r,$$

$$\kappa_0\tilde{x}^\pm_r\kappa_0^{-1} = q^{\mp 1}\tilde{x}^\pm_r, \tag{3.5.2}$$

$$q^{-1}\tilde{x}^+_r\tilde{x}^-_s - \tilde{x}^-_s\tilde{x}^+_r = \frac{\tilde{\psi}^+_{r+s} - \tilde{\psi}^-_{r+s}}{q-q^{-1}}. \tag{3.5.3}$$

The generators $\tilde{x}^+_r$ and $\tilde{x}^-_r$ satisfy also the relations (3.1.8), (3.1.9) and (3.1.10) after substituting $x \to \tilde{x}$. As before, we have

$$\begin{aligned}
\tilde{\Psi}^+(u) &= \sum_{k=0}^{\infty}\tilde{\psi}^+_k u^k = \exp\left((q-q^{-1})\sum_{l=1}^{\infty}\tilde{h}_l u^l\right),\\
\tilde{\Psi}^-(u) &= \sum_{k=0}^{\infty}\tilde{\psi}^-_{-k} u^{-k} = \kappa_0^2\exp\left(-(q-q^{-1})\sum_{l=1}^{\infty}\tilde{h}_{-l} u^l\right).
\end{aligned}$$

There is an isomorphism between the algebras $U_q(A_2^{(2)})$ and $\tilde{U}$ (see [62])

$$U_q(A_2^{(2)}) \simeq \tilde{U} \otimes_{\mathbb{C}[\kappa_0]} \mathbb{C}[\kappa_0,\kappa_0^{-1}].$$

The Drinfeld generators of $U_q(A_2^{(2)})$ are related to the generators of the asymptotic algebra by

$$x_r^+ = \tilde{x}_r^+, \qquad x_r^- = \kappa_0^{-1}\tilde{x}_r^-, \qquad \psi_r^\pm = \kappa_0^{-1}\tilde{\psi}_r^\pm, \qquad K = \kappa_0^{-1}. \tag{3.5.4}$$

Representations of the asymptotic algebra are obtained from the representations of $U_q(A_2^{(2)})$. First we write

$$\begin{aligned} x_r^+ v_{n_1,n_2} &= A(-q^{2(1-n_1)})^r v_{n_1-1,n_2} + B^{-1} q^k \tilde{\beta}^+_{n_1,n_2} q^{-2rn_2+r} v_{n_1,n_2-1}, \\ x_r^- v_{n_1,n_2} &= A^{-1} q^k \tilde{\alpha}^-_{n_1,n_2}(-q^{-2n_1})^r v_{n_1+1,n_2} + B q^{-2rn_2-r} v_{n_1+1,n_2}, \end{aligned}$$

where we already chose $a = q^{-2k}$ and also

$$\beta^+_{n_1,n_2} = q^k \tilde{\beta}^+_{n_1,n_2}, \qquad \alpha^-_{n_1,n_2} = q^k \tilde{\alpha}^-_{n_1,n_2}.$$

Choose $A = 1$ and $B = q^k$, we get

$$x_r^+ v_{n_1,n_2} = (-q^{2(1-n_1)})^r v_{n_1-1,n_2} + \tilde{\beta}^+_{n_1,n_2} q^{-2rn_2+r} v_{n_1,n_2-1}, \tag{3.5.5}$$
$$x_r^- v_{n_1,n_2} = q^k(\tilde{\alpha}^-_{n_1,n_2}(-q^{-2n_1})^r v_{n_1+1,n_2} + q^{-2rn_2-r} v_{n_1+1,n_2}). \tag{3.5.6}$$

From here we can derive the representations of the asymptotic algebra using (3.5.4)

$$\tilde{x}_r^+ v_{n_1,n_2} = (-q^{2(1-n_1)})^r v_{n_1-1,n_2} + \tilde{\beta}^+_{n_1,n_2} q^{-2rn_2+r} v_{n_1,n_2-1} \tag{3.5.7}$$
$$\tilde{x}_r^- v_{n_1,n_2} = \tilde{\alpha}^-_{n_1,n_2} q^{n_1+n_2+1}(-q^{-2n_1})^r v_{n_1+1,n_2} + q^{-2rn_2-r+n_1+n_2+1} v_{n_1+1,n_2}, \tag{3.5.8}$$

Now we can perform the limit $k \to \infty$ and obtain the asymptotic representation on the space $V_\infty = \bigoplus_{0\le n_1\le n_2} v_{n_1,n_2}$

$$\tilde{x}_r^+ v_{i,j} = \left(-q^{2-2i}\right)^r v_{i-1,j} + \frac{q^{i-2jr-j+r+3}\left(q^{2j+1}+1\right)\left(q^{2j}-q^{2i}\right)}{(q-1)^2(q+1)\left(q^{2i+1}+q^{2j}\right)\left(q^{2i}+q^{2j+1}\right)} v_{i,j-1}, \tag{3.5.9}$$
$$\tilde{x}_r^- v_{i,j} = \frac{\left(-q^{-2i}\right)^r q^{2j+3}\left(q^{2i+2}-1\right)\left(q^{2j}-q^{2i}\right)}{(q-1)^2(q+1)\left(q^{2i+1}+q^{2j}\right)\left(q^{2i}+q^{2j+1}\right)} v_{i+1,j} + q^{i-2jr+j-r+1} v_{i,j+1}, \tag{3.5.10}$$
$$\tilde{\Psi}^+(u) v_{i,j} = \frac{q^{2i+2j+1}\left(q^2u+1\right)\left(q^{2i}-qu\right)\left(q^{2j}+u\right)}{\left(q^{2i}+u\right)\left(q^{2i}+q^2u\right)\left(q^{2j+1}-u\right)\left(q^{2j}-qu\right)} v_{i,j}, \tag{3.5.11}$$
$$\tilde{\Psi}^-(u) v_{i,j} = \frac{q^{2i+2j+1}u^{-1}\left(q^2+u^{-1}\right)\left(q^{2i}u^{-1}-q\right)\left(q^{2j}u^{-1}+1\right)}{\left(q^{2i}u^{-1}+1\right)\left(q^{2i}u^{-1}+q^2\right)\left(q^{2j+1}u^{-1}-1\right)\left(q^{2j}u^{-1}-q\right)} v_{i,j}. \tag{3.5.12}$$

Note, that $\tilde{\psi}_0^-$ vanishes in this representation, and $\tilde{\psi}_0^+ = 1$. Let us turn to the Drinfeld–Jimbo presentation. Using Eqs. (3.1.13) and (3.1.14), we obtain an isomorphism between the asymptotic algebra $\tilde{U}$ and the Borel subalgebra $U_q(\mathfrak{b}^+)$

$$\begin{aligned} e_0^+ v_{i,j} &= (q-1)q^{2i+2} v_{i,j+2} + \frac{\left(q^2+1\right)\left(q^{2i+2}-1\right)q^{i+3j+5}}{(q-1)\left(q^{2i+1}+q^{2j}\right)\left(q^{2i}+q^{2j+3}\right)} v_{i+1,j+1} \\ &- \frac{\left(q^{4i+6}-\left(q^2+1\right)q^{2i+2}+1\right)\left(q^{2j}-q^{2i}\right)\left(q^{2j}-q^{2i+2}\right)q^{-2i+4j+5}}{(q-1)^3(q+1)^2\left(q^{2i+1}+q^{2j}\right)^2\left(q^{2i+3}+q^{2j}\right)\left(q^{2i}+q^{2j+1}\right)} v_{i+2,j}, \end{aligned} \tag{3.5.13}$$
$$e_1^+ v_{i,j} = \frac{\left(q^{2j+1}+1\right)\left(q^{2j}-q^{2i}\right)q^{i-j+3}}{(q-1)^2(q+1)\left(q^{2i+1}+q^{2j}\right)\left(q^{2i}+q^{2j+1}\right)} v_{i,j-1} + v_{i-1,j}. \tag{3.5.14}$$

This representation is characterised by the Drinfeld rational fraction, i.e. the highest weight of $\tilde{\Psi}(u)$

$$\tilde{\Psi}_0 = \frac{1}{1 - q^{-1}u}. \tag{3.5.15}$$

One can check that this representation satisfies all relations of the Borel subalgebra. This way of constructing the Borel representations from the asymptotic algebra is due to [62]. For the untwisted algebras it led to some understanding of the appearance of the q-oscillator representations previously found in [5, 4, 81] and [14, 17] for the algebras corresponding to the A series. In Chapter 4 the infinite dimensional representations will be very important for us, however, we will not make use of the Borel subalgebras and their asymptotic representations. Instead, we will work with the "bosonic" modes of the Drinfeld generators.

Looking at the representations above (3.5.13)-(3.5.14) and also (3.5.9)-(3.5.10) one can already anticipate the troubles that we will run into. The problem is that the representation that we wrote depends on rational functions in the variables q^i and q^j. Eventually we would like to take the trace over the auxiliary space, this means we will encounter summations over i and j which run from zero to infinity. Performing such summations when the rational functions are involved is a hard problem. Note, this issue does not appear in the untwisted case.

Chapter 4

Universal $\mathcal{R}$-matrix for $U_q(A_2^{(2)})$

The R-matrices can be constructed using the notion of the universal $\mathcal{R}$-matrix. The universal $\mathcal{R}$-matrix lives in some completion of the tensor product of the quantum affine algebra $\mathcal{A} \otimes \mathcal{A}$. Taking a particular representation of $\mathcal{A}$ we obtain an R-matrix which automatically satisfies the Yang–Baxter equation. The latter is ensured by the definition of the universal $\mathcal{R}$-matrix. We will compute R-matrices using the representations obtained in Chapter 3, but first we need an explicit formula for the universal $\mathcal{R}$-matrix written in terms of the generators of the algebra of our interest $U_q(A_2^{(2)})$. For our purposes it is better to use the Khoroshkin–Tolstoy (KT) construction of the universal $\mathcal{R}$-matrix [117, 73] (see also [91, 32]).

The KT formula is written in terms of the higher root vectors $e_\gamma^{\pm}$ of the algebra. These higher root vectors must obey the Cartan–Weyl condition, i.e. the commutators of the root vectors $[e_\gamma^{+}, e_\gamma^{-}]$ must have a simple form expressed in the Cartan generators. The construction of this basis is given after the preliminaries. Next, we will present the KT formula. The first application of the KT formula will be the calculation of the IK R-matrix. It corresponds to the representation $V^{(1)} \otimes V^{(1)}$. After that we will compute the R-matrix for the representation $V^{(1)} \otimes V^{(k)}$ with the free parameter k which labels the representation spaces of different dimensions. Next, we will send k to infinity in a certain way and obtain an L-matrix (which is the R-matrix for $V^{(1)} \otimes V^{(\infty)}$). This L-matrix corresponds to the representation $V^{(\infty)}$ associated to the Drinfeld rational fraction (4.6.5). There are two more L-matrices: one corresponds to the Drinfeld fraction (3.5.15) and the second one, roughly speaking, to the inverse of the fraction (3.5.15). These two L-matrices have a complicated form, therefore we do not present them here.

Our initial goal was to compute the L-matrix that will allow us to build the Baxter's Q-operator. Under the Baxter's Q-operator, we understand the transfer matrix whose eigenvalues are the generating functions of the Bethe roots. The Baxter's Q-operators were found for $U_q(A_1^{(1)})$ and $U_q(A_2^{(1)})$ in [5, 4] as transfer matrices constructed from the L-matrices. Finding the Baxter's Q-operator for $U_q(A_2^{(2)})$ turns out to be a very complicated problem at this stage. As we mentioned, the L-matrices have an infinite dimensional representation in one of the tensor components. These representations are defined by the Drinfeld rational fractions [62]. We expect to have three different L-

matrices corresponding to three Drinfeld rational functions [1]. It turns out that only one of them has a good form which permits computing the trace (it appeared first in [15]). This L-matrix does not correspond to the Baxter's Q-operator. We made an attempt to compute another L-matrix which corresponds to (3.5.15). The components of the latter L-matrix are very complicated to work with. In particular, it is not evident how to take the trace in order to compute the corresponding Q-operator.

4.1 Introduction

Let us recall the two equations which define the universal $\mathcal{R}$-matrix

$$\Delta'(x) = \mathcal{R}\Delta(x)\mathcal{R}^{-1}, \quad \forall x \in \mathcal{A}, \tag{4.1.1}$$

$$(\Delta \otimes \mathrm{id})\mathcal{R} = \mathcal{R}_{1,3}\mathcal{R}_{2,3}, \quad (\mathrm{id} \otimes \Delta)\mathcal{R} = \mathcal{R}_{1,3}\mathcal{R}_{1,2}. \tag{4.1.2}$$

The first equation here is called the quasi-commutativity condition, while the second one is called the quasi-triangularity condition. If we restrict to the $\mathcal{R}$-matrices of the form

$$\mathcal{R} = \sum_i u_i \otimes v_i,$$

where u_i and v_i are lowering and raising elements of the quantum algebra, then the quasi-commutativity condition is enough to fix the universal $\mathcal{R}$-matrix up to a constant [68, 73]. Khoroshkin and Tolstoy [117] obtained a formula for the universal $\mathcal{R}$-matrix and then showed that it satisfies the relation (4.1.1) applied to all generators of the algebra. The second statement follows from a certain property of the q-exponentials [72]. The definition of the q-exponentials is given in (4.3.4). The main theorem of [73] says that there is a unique universal $\mathcal{R}$-matrix written in the Ansatz form

$$\mathcal{R} = \tilde{\mathcal{R}}\mathcal{K}, \quad \tilde{\mathcal{R}} \in T_q(U_q(\mathfrak{b}^+) \otimes U_q(\mathfrak{b}^-)), \tag{4.1.3}$$

where $\mathcal{K}$ is a simple factor constructed from the Cartan elements while the reduced matrix $\tilde{\mathcal{R}}$ lives in the Taylor extension of the tensor product of the Borel subalgebras $T_q(U_q(\mathfrak{b}^+) \otimes U_q(\mathfrak{b}^-))$. The Taylor extension stands for the linear space of formal Taylor series consisting of monomials in $e_\gamma^\pm$ with coefficients being rational functions in the Cartan elements. More explicitly, the reduced matrix $\tilde{\mathcal{R}}$ is a product of q-exponentials with exponents of the form const $\times\, e_\gamma^+ \otimes e_\gamma^-$, and the product is running over the root system. In order to make sense out of this product one defines the normal ordering of the roots and then defines the associated higher root vectors. The higher root vectors are certain products of simple root vectors. In Section 4.2 we write the normal ordering and the higher root vectors in the Cartan–Weyl basis. The normal ordering of the roots for $U_q(A_2^{(2)})$ was given in the paper [73]. There, also the Cartan–Weyl basis is proposed. However, we prefer to use the Drinfeld generators, hence we redo the computation in our conventions and show the Cartan–Weyl property of the constructed higher roots.

1. The three rational fractions correspond to the three choices of the reference vectors $v_{0,0}$, $v_{0,k}$ and $v_{k,k}$ on the diagram (3.3), where k is assumed to be sent to infinity.

4.2 Cartan–Weyl basis

Before we write the Cartan–Weyl basis for the algebra $U_q(A_2^{(2)})$ let us make a couple of redefinitions. First of all, we rescale the generators

$$e_0^{\pm} \to \sqrt{q+q^{-1}}\; e_0^{\pm}, \quad e_1^{\pm} \to \frac{1}{\sqrt{q^{1/2}+q^{-1/2}}}\; e_1^{\pm}. \tag{4.2.1}$$

This rescaling affects the definition of the algebra, namely, the non homogeneous relation (3.1.3) has to be rewritten as

$$[e_i^+, e_j^-] = \delta_{i,j}\frac{k_i - k_i^{-1}}{q-q^{-1}}. \tag{4.2.2}$$

This relation remains the same if we rescale the generators in the following way

$$e_i^+ \to a_i\, e_i^+, \quad e_i^- \to a_i^{-1} e_i^-, \tag{4.2.3}$$

with some coefficients a_i. Our formulae become nicer if we choose

$$a_1 = 1, \quad a_0 = \frac{1}{q^{3/2}(q+1)[2]_q}. \tag{4.2.4}$$

Now we turn to the normal ordering of the root system. The normal ordering of the roots for $U_q(A_2^{(2)})$ algebra was written in [73]. The root system Δ consists of positive and negative roots $\Delta = \Delta_+ \sqcup \Delta_-$. The root vectors $e_0^{\pm}$ and $e_1^{\pm}$ we rewrite as follows

$$e_i^+ = e_{\alpha_i}, \quad e_i^- = f_{\alpha_i},$$

where α_0 and α_1 are the simple roots of the algebra $U_q(A_2^{(2)})$. The symmetrized Cartan matrix C^s defines the bilinear form on the space dual to the Cartan subalgebra

$$(\alpha_i,\alpha_j) = C^s_{i,j}.$$

Thus we have

$$(\alpha_0,\alpha_0) = 4, \quad (\alpha_1,\alpha_1) = 1, \quad (\alpha_0,\alpha_1) = -2.$$

It is convenient to use δ and α instead

$$\delta = \alpha_0 + 2\alpha_1, \quad \alpha = \alpha_1,$$

then the bilinear form becomes

$$(\delta,\delta) = (\alpha,\delta) = 0, \quad (\alpha,\alpha) = 1.$$

With these notations the positive roots can be written as a union of five parts

$$\begin{aligned}\Delta_+ &= \{\alpha + m\delta | m \in \mathbb{Z}_{\geq 0}\} \cup \{2\alpha + (2m+1)\delta | m \in \mathbb{Z}_{\geq 0}\} \\ &\cup \{m\delta | m \in \mathbb{Z}_{>0}\} \cup \{\delta - 2\alpha + 2m\delta | m \in \mathbb{Z}_{\geq 0}\} \cup \{\delta - \alpha + m\delta | m \in \mathbb{Z}_{\geq 0}\}\end{aligned} \tag{4.2.5}$$

The negative roots are defined through $\Delta_- = -\Delta_+$. Usually one uses the following normal ordering on this root system

$$\gamma + k\delta \prec m\delta \prec (\delta-\gamma) + l\delta, \qquad \gamma = \alpha, 2\alpha, \quad k,l \in \mathbb{Z}_{\geq 0},\ m \in \mathbb{Z}_{>0}. \tag{4.2.6}$$

More explicitly, the first set is

$$\alpha,\ 2\alpha+\delta,\ \alpha+\delta,\ 2\alpha+3\delta,\ \alpha+2\delta,\ 2\alpha+5\delta,\ \alpha+3\delta,\ 2\alpha+7\delta,\ \ldots, \tag{4.2.7}$$

the middle set is

$$\delta,\ 2\delta,\ 3\delta,\ 4\delta,\ 5\delta,\ 6\delta,\ \ldots, \tag{4.2.8}$$

and the last set is

$$\ldots,\ \delta-2\alpha+6\delta,\ \delta-\alpha+2\delta,\ \delta-2\alpha+4\delta,\ \delta-\alpha+\delta,\ \delta-2\alpha+2\delta,\ \delta-\alpha,\ \delta-2\alpha. \tag{4.2.9}$$

In our calculations, however, we use another normal ordering, which is the opposite to the above

$$(\delta-\gamma) + l\delta \prec m\delta \prec \gamma + k\delta, \qquad \gamma = \alpha, 2\alpha, \quad k,l \in \mathbb{Z}_{\geq 0},\ m \in \mathbb{Z}_{>0}. \tag{4.2.10}$$

The Cartan–Weyl basis allows us to write the higher root vectors as commutators of simple roots. In this basis we have

$$[e_\gamma, f_\gamma] = \frac{k_\gamma - k_\gamma^{-1}}{q - q^{-1}}, \tag{4.2.11}$$

where $k_\gamma = \prod_i k_i^{m_i}$ if $\gamma = \sum_i m_i \alpha_i$. Now we present the Cartan–Weyl basis and show that the Cartan–Weyl property (4.2.11) holds. In order to do that we will re-express the higher roots in terms of the Drinfeld generators. This was done for untwisted affine Lie algebras in [74]. The positive root vectors corresponding to the higher roots are defined as

$$e_{\delta-\alpha} = [2]_q^{-1/2}[e_{\delta-2\alpha}, e_\alpha]_q, \quad e'_\delta = [e_{\delta-\alpha}, e_\alpha]_q, \tag{4.2.12}$$

$$e_{\alpha+m\delta} = \frac{1}{q+q^{-1}+1}[e'_\delta, e_{\alpha+(m-1)\delta}]_q, \tag{4.2.13}$$

$$e_{\delta-\alpha+m\delta} = \frac{1}{q+q^{-1}+1}[e'_\delta, e_{\delta-\alpha+(m-1)\delta}]_q, \tag{4.2.14}$$

$$e_{2\alpha+(2m-1)\delta} = [2]_q^{-1/2}[e_{\alpha+m\delta}, e_{\alpha+(m-1)\delta}]_q, \tag{4.2.15}$$

$$e_{\delta-2\alpha+2(m+1)\delta} = [2]_q^{-1/2}[e_{\delta-\alpha+m\delta}, e_{\delta-\alpha+(m+1)\delta}]_q, \tag{4.2.16}$$

$$e'_{m\delta} = [e_{\delta-\alpha}, e_{\alpha+(m-1)\delta}]_q. \tag{4.2.17}$$

The negative root vectors are given by

$$f_{\delta-\alpha} = [2]_q^{-1/2}[f_\alpha,f_{\delta-2\alpha}]_{q^{-1}}, \quad f'_\delta = [f_\alpha,f_{\delta-\alpha}]_{q^{-1}}, \tag{4.2.18}$$

$$f_{\alpha+m\delta} = \frac{1}{q+q^{-1}+1}[f_{\alpha+(m-1)\delta},f'_\delta]_{q^{-1}}, \tag{4.2.19}$$

$$f_{\delta-\alpha+m\delta} = \frac{1}{q+q^{-1}+1}[f_{\delta-\alpha+(m-1)\delta},f'_\delta]_{q^{-1}}, \tag{4.2.20}$$

$$f_{2\alpha+(2m-1)\delta} = [2]_q^{-1/2}[f_{\alpha+(m-1)\delta},f_{\alpha+m\delta}]_{q^{-1}}, \tag{4.2.21}$$

$$f_{\delta-2\alpha+2(m+1)\delta} = [2]_q^{-1/2}[f_{\delta-\alpha+(m+1)\delta}, f_{\delta-\alpha+m\delta}]_{q^{-1}}, \tag{4.2.22}$$

$$f'_{m\delta} = [f_{\alpha+(m-1)\delta},f_{\delta-\alpha}]_{q^{-1}}. \tag{4.2.23}$$

We used here the q-commutator

$$[e_\gamma,e_{\gamma'}]_q = e_\gamma e_{\gamma'} - q^{(\gamma,\gamma')}e_{\gamma'}e_\gamma. \tag{4.2.24}$$

Following [73] we also define the unprimed imaginary root vectors

$$e_\delta(u) = \frac{1}{(q^{1/2}-q^{-1/2})}\log(1+(q-q^{-1})e'_\delta(u)) \tag{4.2.25}$$

$$f_\delta(u) = \frac{-1}{(q^{1/2}-q^{-1/2})}\log(1-(q-q^{-1})f'_\delta(u)) \tag{4.2.26}$$

Where we used the generating functions $e_\delta(u)$, $f_\delta(u)$ and $e'_\delta(u)$, $f'_\delta(u)$ for the unprimed and primed imaginary root vectors respectively

$$e_\delta(u) = \sum_{m>0} u^m e_{m\delta}, \quad e'_\delta(u) = \sum_{m>0} u^m e'_{m\delta}, \tag{4.2.27}$$

$$f_\delta(u) = \sum_{m>0} u^{-m} f_{m\delta}, \quad f'_\delta(u) = \sum_{m>0} u^{-m} f'_{m\delta}. \tag{4.2.28}$$

Let us show that the roots written in this form are Cartan–Weyl. For this purpose we will use the Drinfeld presentation in which we take into account the redefinition (3.1.12) and omit the bar in $\bar{x}_r^\pm$ and $\bar{h}_r$ for simplicity. From now on we will work with the redefined in this way Drinfeld generators. First we find that

$$e_\alpha = [2]_{q^{1/2}}^{-1/2}x_0^+, \quad f_\alpha = [2]_{q^{1/2}}^{-1/2}x_0^- \tag{4.2.29}$$

Let us write Eq. (4.2.12) in which we substitute (3.1.13) and (3.1.14) to pass to the Drinfeld generators

$$\begin{aligned} e_{\delta-\alpha} = [2]_q^{-1/2}[e_{\delta-2\alpha},e_\alpha]_q = \frac{1}{q^{3/2}(q+1)[2]_q[2]_{q^{1/2}}^{1/2}} \\ \times K^{-2}(x_0^-x_1^-x_0^+ - qx_1^-x_0^-x_0^+ - x_0^+x_0^-x_1^- + qx_0^+x_1^-x_0^-), \end{aligned} \tag{4.2.30}$$

where we used

$$x_r^\pm K^{-1} = q^{\pm 1}K^{-1}x_r^\pm.$$

Now we commute x_0^+ to the left using Eq. (3.1.6), after cancelations we get

$$e_{\delta-\alpha} = \frac{1}{q^{3/2}(q+1)[2]_q[2]_{q^{1/2}}^{1/2}} K^{-2}\Big((q+1)\sqrt{q}Kx_1^- + qK[h_1,x_0^-]\Big), \tag{4.2.31}$$

where we took into account the relation between ψ_1 and h_1 that can be obtained from (3.1.11)

$$\psi_1 = (q^{1/2} - q^{-1/2})Kh_1, \tag{4.2.32}$$

and for the later use we add here also

$$\psi_{-1} = -(q^{1/2} - q^{-1/2})K^{-1}h_{-1}. \tag{4.2.33}$$

In the last term in the brackets in (4.2.31) we use (3.1.7) (keeping in mind (3.1.12))

$$e_{\delta-\alpha} = -q^{-1}[2]_{q^{1/2}}^{-1/2}K^{-1}x_1^-. \tag{4.2.34}$$

A similar calculation for the negative root vectors gives

$$f_{\delta-\alpha} = -[2]_{q^{1/2}}^{-1/2}Kx_{-1}^+. \tag{4.2.35}$$

If we take the commutator of these two root vectors, we get

$$\begin{aligned}[e_{\delta-\alpha},f_{\delta-\alpha}] &= q^{-1}[2]_{q^{1/2}}^{-1}[K^{-1}x_1^-,Kx_{-1}^+] \\ &= -[2]_{q^{1/2}}^{-1}[x_{-1}^+,x_1^-] = \frac{K^{-1}-K}{q-q^{-1}},\end{aligned}$$

as it should be since $k_{\delta-\alpha} = k_1k_0 = K^{-1}$ and the result agrees with (4.2.11). Next we compute the imaginary root vectors e'_δ and f'_δ using Eqs. (4.2.34), (4.2.35) and (4.2.29) and the commutation relations of the Drinfeld generators. This calculation yields

$$e'_\delta = q^{-1}[2]_{q^{1/2}}^{-1}h_1, \quad f'_\delta = q[2]_{q^{1/2}}^{-1}h_{-1} \tag{4.2.36}$$

Looking at Eq. (4.2.29) we notice that the recursive formulae (4.2.13) and (4.2.19) become similar to Eqs. (3.1.7). We can solve the recurrence relations given by (4.2.13) and (4.2.19) in terms of the Drinfeld generators

$$e_{\alpha+m\delta} = q^{-m}[2]_{q^{1/2}}^{-1/2}x_m^+, \quad f_{\alpha+m\delta} = q^m[2]_{q^{1/2}}^{-1/2}x_{-m}^-. \tag{4.2.37}$$

The Cartan–Weylness (4.2.11) is easily verified

$$[e_{\alpha+m\delta},f_{\alpha+m\delta}] = [2]_{q^{1/2}}^{-1}[x_m^+,x_{-m}^-] = \frac{K-K^{-1}}{q-q^{-1}}. \tag{4.2.38}$$

The Cartan element of the vector $\alpha+m\delta$ is $k_1^{2m+1}k_0^m = K$, hence the right hand side. Now we solve the recurrence relations given by (4.2.14) and (4.2.20)

$$e_{\delta-\alpha+m\delta} = (-q)^{-1-m}[2]_{q^{1/2}}^{-1/2}K^{-1}x_{m+1}^-, \quad f_{\delta-\alpha+m\delta} = -(-q)^m[2]_{q^{1/2}}^{-1/2}Kx_{-m-1}^+. \tag{4.2.39}$$

It is easy to check that these two vectors have the commutator

$$[e_{\delta-\alpha+m\delta}, f_{\delta-\alpha+m\delta}] = (-q)^{-1}[2]^{-1}_{q^{1/2}}[K^{-1}x^-_{m+1}, Kx^+_{-m-1}]$$
$$= -[2]^{-1}_{q^{1/2}}[x^+_{-m-1}, x^-_{m+1}] = \frac{K^{-1}-K}{q-q^{-1}}. \tag{4.2.40}$$

The Cartan element is $k_{\delta-\alpha+m\delta} = k_1^{2m+1}k_0^{m+1} = K^{-1}$, as expected.

Plugging the roots $e_{\alpha+m\delta}$ and $f_{\alpha+m\delta}$ from (4.2.37) into Eqs. (4.2.15) and (4.2.21), respectively, we obtain

$$e_{2\alpha+(2m-1)\delta} = [2]^{-1}_{q^{1/2}}[2]_q^{-1/2}q^{-2m+1}(x^+_m x^+_{m-1} - qx^+_{m-1}x^+_m), \tag{4.2.41}$$
$$f_{2\alpha+(2m-1)\delta} = [2]^{-1}_{q^{1/2}}[2]_q^{-1/2}q^{2m-1}(x^-_{-m+1}x^-_{-m} - q^{-1}x^-_{-m}x^-_{-m+1}). \tag{4.2.42}$$

The Cartan–Weyl condition is

$$[e_{2\alpha+(2m-1)\delta}, f_{2\alpha+(2m-1)\delta}] = \frac{K^2-K^{-2}}{q-q^{-1}}.$$

One can check that it holds by repetitive application of the commutation relations (3.1.5)-(3.1.7) and also using the relation between ψ_1 and h_1 and ψ_{-1} and h_{-1} (4.2.32)-(4.2.33). Similarly, we compute (4.2.16) and (4.2.22) and find

$$e_{\delta-2\alpha+2(m+1)\delta} = -[2]^{-1}_{q^{1/2}}[2]_q^{-1/2}q^{-2m-4}K^{-2}(x^-_{m+1}x^-_{m+2} - qx^-_{m+2}x^-_{m+1}), \tag{4.2.43}$$
$$f_{\delta-2\alpha+2(m+1)\delta} = -[2]^{-1}_{q^{1/2}}[2]_q^{-1/2}q^{2m}K^2(x^+_{-m-2}x^+_{-m-1} - q^{-1}x^+_{-m-1}x^+_{-m-2}). \tag{4.2.44}$$

The Cartan–Weyl condition reads

$$[e_{\delta-2\alpha+2(m+1)\delta}, f_{\delta-2\alpha+2(m+1)\delta}] = \frac{K^{-2}-K^2}{q-q^{-1}},$$

and can be verified by application of the commutation relations of the Drinfeld generators as previously. Let us turn to the primed imaginary root vectors (4.2.17) and (4.2.23). Substituting (4.2.34) and the e-formula from (4.2.37) into (4.2.17) we get

$$e'_{m\delta} = [e_{\delta-\alpha}, e_{\alpha+(m-1)\delta}]_q = -q^{-m}[2]^{-1}_{q^{1/2}}K^{-1}[x^-_1, x^+_{m-1}]$$
$$= q^{-m}[2]^{-1}_{q^{1/2}}K^{-1}\frac{\psi^+_m - \psi^-_m}{q^{1/2}-q^{-1/2}}.$$

Similarly, substituting (4.2.35) and the f-formula from (4.2.37) into (4.2.23), we get

$$f'_{m\delta} = [f_{\alpha+(m-1)\delta}, f_{\delta-\alpha}]_{q^{-1}} = q^m[2]^{-1}_{q^{1/2}}K[x^+_{-1}, x^-_{-m+1}]$$
$$= q^m[2]^{-1}_{q^{1/2}}K\frac{\psi^+_{-m} - \psi^-_{-m}}{q^{1/2}-q^{-1/2}}.$$

For $m > 0$, $\psi^-_m = \psi^+_{-m} = 0$, thus we obtain

$$e'_{m\delta} = q^{-m}K^{-1}\frac{\psi^+_m}{q-q^{-1}}, \tag{4.2.45}$$

$$f'_{m\delta} = -q^m K\frac{\psi^-_{-m}}{q-q^{-1}}. \tag{4.2.46}$$

Finally we write the unprimed imaginary root vectors. Inserting the coefficients (4.2.45) and (4.2.46) in the corresponding generating functions in (4.2.27) and (4.2.28) and recalling the relation between ψ_m and h_m (3.1.11) we find

$$
\begin{aligned}
e'(u) &= \frac{1}{q-q^{-1}}\left(e^{(q^{1/2}-q^{-1/2})\sum_{m>0} h_m u^m q^{-m}} - 1\right),\\
f'(u) &= \frac{-1}{q-q^{-1}}\left(e^{-(q^{1/2}-q^{-1/2})\sum_{m>0} h_{-m} u^{-m} q^{m}} - 1\right).
\end{aligned}
$$

Plugging this into (4.2.27) and (4.2.28) we get

$$
e_{m\delta} = q^{-m} h_m, \tag{4.2.47}
$$
$$
f_{m\delta} = q^{m} h_{-m}. \tag{4.2.48}
$$

This concludes the construction of the Cartan–Weyl basis. All higher roots are now expressed through the Drinfeld generators, which is very convenient as we already know how the Drinfeld generators act on the representation $V^{(k)}$. Let us summarize the results of this section. We list all higher roots written in terms of the Drinfeld generators.

$$
\begin{aligned}
&e_{\alpha+m\delta} = q^{-m}[2]_{q^{1/2}}^{-1/2} x_m^+, \quad f_{\alpha+m\delta} = q^{m}[2]_{q^{1/2}}^{-1/2} x_{-m}^-,\\
&e_{\delta-\alpha+m\delta} = (-q)^{-1-m}[2]_{q^{1/2}}^{-1/2} K^{-1} x_{m+1}^-, \quad f_{\delta-\alpha+m\delta} = -(-q)^{m}[2]_{q^{1/2}}^{-1/2} K x_{-m-1}^+,\\
&e_{2\alpha+(2m-1)\delta} = [2]_{q^{1/2}}^{-1}[2]_q^{-1/2} q^{-2m+1}(x_m^+ x_{m-1}^+ - q x_{m-1}^+ x_m^+),\\
&f_{2\alpha+(2m-1)\delta} = [2]_{q^{1/2}}^{-1}[2]_q^{-1/2} q^{2m-1}(x_{-m+1}^- x_{-m}^- - q^{-1} x_{-m}^- x_{-m+1}^-),\\
&e_{\delta-2\alpha+2m\delta} = -[2]_{q^{1/2}}^{-1}[2]_q^{-1/2} q^{-2m-2} K^{-2}(x_m^- x_{m+1}^- - q x_{m+1}^- x_m^-),\\
&f_{\delta-2\alpha+2m\delta} = -[2]_{q^{1/2}}^{-1}[2]_q^{-1/2} q^{2m-2} K^{2}(x_{-m-1}^+ x_{-m}^+ - q^{-1} x_{-m}^+ x_{-m-1}^+),\\
&e'_{m\delta} = q^{-m} K^{-1}\frac{\psi_m^+}{q-q^{-1}}, \quad f'_{m\delta} = -q^{m} K \frac{\psi_{-m}^-}{q-q^{-1}},\\
&e_{m\delta} = q^{-m} h_m, \quad f_{m\delta} = q^{m} h_{-m}.
\end{aligned}
\tag{4.2.49}
$$

Now we can turn to the KT formula and the construction of the R-matrices.

4.3 Khoroshkin–Tolstoy formula

The universal $\mathcal{R}$-matrix of KT is constructed as a product of four terms

$$
\mathcal{R} = \mathcal{R}_{\succ\delta}\mathcal{R}_{\delta}\mathcal{R}_{\prec\delta}\mathcal{K}. \tag{4.3.1}
$$

This formula is the same as previously (3.2.5), except that we use a different indexing of the factors to reflect the ordering of the higher roots. First factor is defined as

$$
\mathcal{R}_{\succ\delta} = \prod_m \mathcal{R}_{\delta-\alpha,m} \prod_m \mathcal{R}_{\delta-2\alpha,m}, \tag{4.3.2}
$$
$$
\mathcal{R}_{\delta-\gamma,m} = \exp_{q_\gamma}((q-q^{-1}) e_{\delta-\gamma+m\delta} \otimes f_{\delta-\gamma+m\delta}), \tag{4.3.3}
$$

where $\gamma = \alpha$, 2α, $m \in \mathbb{Z}_{\geq 0}$ and the order of the products coincides with the chosen normal ordering. We used the notation $q_\gamma = q^{-(\gamma,\gamma)}$, and the q-exponential is understood as the series

$$\exp_q(x) = 1 + x + \frac{x^2}{(2)_q!} + \frac{x^3}{(3)_q!} + .., \tag{4.3.4}$$

where

$$(n)_q! = (n)_q(n-1)_q..(2)_q(1)_q, \qquad (n)_q = \frac{q^n - 1}{q - 1}.$$

The term $\mathcal{R}_\delta$ in our conventions becomes

$$\mathcal{R}_\delta = \exp\left((q^{1/2} - q^{-1/2}) \sum_{m>0} \frac{m}{[2m]_{q^{1/2}}(q^m + q^{-m} + (-1)^{m+1})}\; e_{m\delta} \otimes f_{m\delta}\right). \tag{4.3.5}$$

The third factor is again a product of q-exponentials

$$\mathcal{R}_{\prec\delta} = \prod_m \mathcal{R}_{\alpha,m} \prod_m \mathcal{R}_{2\alpha,m}, \tag{4.3.6}$$

$$\mathcal{R}_{\gamma,m} = \exp_{q_\gamma}((q - q^{-1}) e_{\gamma+m\delta} \otimes f_{\gamma+m\delta}). \tag{4.3.7}$$

where $\gamma = \alpha$, 2α, $m \in \mathbb{Z}_{\geq 0}$ and the order of the products coincides with the chosen normal ordering. Finally, the last term is defined as follows. For any two vectors $v \in V^{(k_1)}$ and $w \in V^{(k_2)}$ we have $kv = q^{(\lambda,\alpha)}v$ and $kw = q^{(\mu,\alpha)}w$, where λ and μ are the weights of v and w, then the matrix elements of $\mathcal{K}$ in the space $V^{(k_1)} \otimes V^{(k_2)}$ are defined as

$$\mathcal{K} v \otimes w = q^{(\lambda,\mu)} v \otimes w. \tag{4.3.8}$$

4.4 Khoroshkin–Tolstoy formula and $V^{(1)} \otimes V^{(1)}$

In this section we use the Khoroshkin–Tolstoy (KT) formula [117] in order to compute the R-matrix for the tensor product of two lowest dimensional representations $(k = 1)$ $V^{(1)}_{\zeta_1} \otimes V^{(1)}_{\zeta_2}$. This calculation was first done in [73], see also [15].

In order to proceed with the formula (4.3.1) we need to specify the representation space. As will be seen below it will allow us to truncate the series of $\exp_q(x)$ and thus we can perform further computations. In this section we will work with the first lowest dimensional representation space $V^{(1)}_\zeta$, where ζ is the spectral parameter attached to the representation space $V^{(1)}_\zeta$, and the corresponding representation map will be called φ^1_ζ. As the R-matrix belongs to $U_q(A^{(2)}_2) \otimes U_q(A^{(2)}_2)$ we will consider its representation on the space $V^{(1)}_{\zeta_1} \otimes V^{(1)}_{\zeta_2}$. We write $\varphi^1_\zeta(U_q(A^{(2)}_2))$ in the matrix form taking into account the redefined conventions from the beginning of Section 4.3

$$\varphi^1_\zeta(f_0) = [2]^{1/2} \begin{pmatrix} 0 & 0 & \zeta^{-1} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \varphi^1_\zeta(e_0) = [2]^{1/2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \zeta & 0 & 0 \end{pmatrix}, \tag{4.4.1}$$

$$\varphi^1_\zeta(f_1) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \varphi^1_\zeta(e_1) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}. \tag{4.4.2}$$

The Drinfeld generators in the matrix form read

$$\varphi^1_\zeta(x_r^+) = [2]_{q^{1/2}}^{1/2} \begin{pmatrix} 0 & q^{-r}\zeta^r & 0 \\ 0 & 0 & (-1)^r\zeta^r \\ 0 & 0 & 0 \end{pmatrix},$$
$$\varphi^1_\zeta(x_r^-) = [2]_{q^{1/2}}^{1/2} \begin{pmatrix} 0 & 0 & 0 \\ q^{-r}\zeta^r & 0 & 0 \\ 0 & (-1)^r\zeta^r & 0 \end{pmatrix}, \tag{4.4.3}$$

and h_r is

$$\varphi^1_\zeta(h_r) = \begin{pmatrix} \frac{q^{\frac{1}{2}-3r}(q^{2r}-1)\zeta^r}{(q-1)r} & 0 & 0 \\ 0 & \frac{(-1)^{1-r}q^{\frac{1}{2}-2r}(q^{2r}-1)((-1)^r q^r-1)\zeta^r}{(q-1)r} & 0 \\ 0 & 0 & \frac{\sqrt{q}(-1)^{r+1}(q^{2r}-1)\zeta^r}{(q-1)r} \end{pmatrix}. \tag{4.4.4}$$

Let us start with the middle term $R_\delta = \varphi^1_{\zeta_1} \otimes \varphi^1_{\zeta_2} \circ \mathcal{R}_\delta$, rewritten in the Drinfeld generators

$$R_\delta = \exp\left((q^{1/2} - q^{-1/2}) \sum_{r>0} \frac{r}{[2r]_{q^{1/2}}(q^r + q^{-r} + (-1)^{r+1})} \varphi^1_{\zeta_1}(h_r) \otimes \varphi^1_{\zeta_2}(h_{-r}) \right). \tag{4.4.5}$$

This term is clearly diagonal. Since the Yang–Baxter equation is homogeneous in R we can choose the normalization of the R-matrix. It is convenient already at this point to choose the normalization. Denote it by $\rho_1(\zeta_1/\zeta_2)$ and write for simplicity ζ_{12} instead of ζ_1/ζ_2. We make the following choice

$$(\varphi^1_{\zeta_1} \otimes \varphi^1_{\zeta_2} \circ \mathcal{R}_\delta).v_0 \otimes v_0 = \rho_1(\zeta_{12}) v_0 \otimes v_0, \tag{4.4.6}$$

with

$$\rho_1(\zeta) = \exp\left(\sum_{r=1}^{\infty} \frac{(q^r - q^{-r})\,\zeta^r}{r\,(q^{-r} + q^r + (-1)^{r+1})} \right), \tag{4.4.7}$$

which can be easily verified by substituting (4.4.4) into (4.4.5) and performing the computation in (4.4.6). Therefore, now we need to compute the following matrix

$$\bar{R}_\delta(\zeta_{12}) = (\varphi^1_{\zeta_1} \otimes \varphi^1_{\zeta_2} \circ \mathcal{R}_\delta)(\rho_1(\zeta_{12}) I)^{-1}, \tag{4.4.8}$$

where I is the nine dimensional identity matrix. After using the matrix representation of h_r (4.4.4) the matrix elements of $\bar{R}_\delta$ become of the form

$$\exp\left(\sum_i \sum_{r=1}^{\infty} \frac{a_i^r}{r} - \sum_j \sum_{r=1}^{\infty} \frac{b_j^r}{r} \right) = \frac{\prod_i (1-a_i)}{\prod_j (1-b_j)}, \tag{4.4.9}$$

with a_i,$b_j \in \mathbb{C}$ and i and j running over some finite sets of integers. This property is expected to hold for any $V^{(k)}$ representation. Using the matrix units $\epsilon_{i,j}$ we can write the "imaginary" part of the R-matrix as

$$\bar{R}_\delta = \epsilon_{1,1} \otimes \epsilon_{1,1} + \frac{(1-\zeta_{12})\, q^2 \epsilon_{1,1} \otimes \epsilon_{2,2}}{q^2 - \zeta_{12}} + \frac{(1-\zeta_{12})\, q^4\, (\zeta_{12}+q)\, \epsilon_{1,1} \otimes \epsilon_{3,3}}{(q^2-\zeta_{12})\,(\zeta_{12}+q^3)} + \frac{(1-\zeta_{12}q^2)\, \epsilon_{2,2} \otimes \epsilon_{1,1}}{1-\zeta_{12}} + \frac{q\,(\zeta_{12}q+1)\, \epsilon_{2,2} \otimes \epsilon_{2,2}}{\zeta_{12}+q} + \frac{(1-\zeta_{12})\, q^2 \epsilon_{2,2} \otimes \epsilon_{3,3}}{q^2-\zeta_{12}} + \frac{(1-\zeta_{12}q^2)\,(\zeta_{12}q^3+1)\, \epsilon_{3,3} \otimes \epsilon_{1,1}}{(1-\zeta_{12})\,(\zeta_{12}q+1)} + \frac{(1-\zeta_{12}q^2)\, \epsilon_{3,3} \otimes \epsilon_{2,2}}{1-\zeta_{12}} + \epsilon_{3,3} \otimes \epsilon_{3,3}. \tag{4.4.10}$$

Now turn to the computation of the term $\mathcal{R}_{\prec\delta}$ which consists of the α-term $\mathcal{R}_\alpha$ and the 2α-term $\mathcal{R}_{2\alpha}$ as in (4.3.6). The q-exponential series of $\mathcal{R}_{\alpha,m}$ truncates after the third term because of the nilpotency of the operators e and f in $k=1$ representation. Thus we get

$$R_\alpha(\zeta_{12}) = (\varphi^1_{\zeta_1} \otimes \varphi^1_{\zeta_2} \circ \mathcal{R}_\alpha) = \varphi^1_{\zeta_1} \otimes \varphi^1_{\zeta_2} \circ \prod_{m\geq 0} (1 + t_{\alpha,m} + \frac{t^2_{\alpha,m}}{(2)_{q_\alpha}!}), \tag{4.4.11}$$

where

$$t_{\alpha,m} = (q-q^{-1}) e_{\alpha+m\delta} \otimes f_{\alpha+m\delta} = (q-q^{-1})[2]^{-1}_{q^{1/2}} x^+_m \otimes x^-_{-m}. \tag{4.4.12}$$

Once again, because of the nilpotency of the operators, the product in (4.4.11) truncates and we get

$$R_\alpha(\zeta_{12}) = \varphi^1_{\zeta_1} \otimes \varphi^1_{\zeta_2} \circ \left(1 + \sum_{m\geq 0} t_{\alpha,m} + \sum_{m\geq 0}\sum_{n\geq 0} t_{\alpha,m+n+1} t_{\alpha,m} + \frac{q}{q+1} \sum_{m\geq 0} t^2_{\alpha,m}\right) \tag{4.4.13}$$

Note that the order of the terms in the second summation appears according to the chosen order of the roots. After computing each term and gathering the results together one finds

$$R_\alpha(\zeta_{12}) = I + \frac{1-q^2}{(\zeta_{12}-1)\,q}(\epsilon_{1,2} \otimes \epsilon_{2,1} + \epsilon_{2,3} \otimes \epsilon_{3,2}) \frac{q^2-1}{\zeta_{12}+q} \epsilon_{2,2} \otimes \epsilon_{1,2} + \frac{q^2-1}{q\,(\zeta_{12}q+1)} \epsilon_{2,3} \otimes \epsilon_{2,1} + - \frac{(q-1)^2 (q+1)\,(q^2-\zeta_{12})}{(\zeta^2_{12}-1)\, q^2\,(\zeta_{12}+q)} \epsilon_{1,3} \otimes \epsilon_{3,1}. \tag{4.4.14}$$

The q-exponential series of $\mathcal{R}_{2\alpha,m}$ truncates after the second term because it is quadratic in e and f in $k=1$ representation. We obtain

$$R_{2\alpha}(\zeta_{12}) = (\varphi^1_{\zeta_1} \otimes \varphi^1_{\zeta_2} \circ \mathcal{R}_{2\alpha}) = \varphi^1_{\zeta_1} \otimes \varphi^1_{\zeta_2} \circ \prod_{m\geq 0} (1 + t_{2\alpha,m}), \tag{4.4.15}$$

where now $t_{2\alpha,m}$ stands for

$$t_{2\alpha,m} = (q-q^{-1}) e_{2\alpha+(2m+1)\delta} \otimes f_{2\alpha+(2m+1)\delta} = (q-q^{-1})[2]^{-2}_{q^{1/2}} [2]^{-1}_q \times (x^+_{m+1} x^+_m - q x^+_m x^+_{m+1}) \otimes (x^-_{-m} x^-_{-m-1} - q^{-1} x^-_{-m-1} x^-_{-m}). \tag{4.4.16}$$

Expanding the product in (4.4.15)

$$R_{2\alpha}(\zeta_{12}) = \varphi^1_{\zeta_1} \otimes \varphi^1_{\zeta_2} \circ \left(1 + \sum_{m\geq 0} t_{2\alpha,m}\right)., \tag{4.4.17}$$

and performing the summation we get

$$R_{2\alpha}(\zeta_{12}) = I - \frac{\zeta_{12}(q^4-1)}{(\zeta_{12}^2-1)\,q^2}\epsilon_{1,3}\otimes\epsilon_{3,1}. \tag{4.4.18}$$

The term $\mathcal{R}_{\succ\delta}$ is computed similarly, it contains the product over the factors $\mathcal{R}_{\delta-\alpha,m}$ which is denoted $\mathcal{R}_{\delta-\alpha}$ and the -2α term $\mathcal{R}_{\delta-2\alpha,m}$ denoted $\mathcal{R}_{\delta-2\alpha}$. The first term is

$$R_{-\alpha}(\zeta_{12}) = (\varphi^1_{\zeta_1} \otimes \varphi^1_{\zeta_2} \circ \mathcal{R}_{\delta-\alpha}) = \varphi^1_{\zeta_1} \otimes \varphi^1_{\zeta_2} \circ \prod_{m\geq 0}(1 + t_{-\alpha,m} + \frac{t^2_{-\alpha,m}}{(2)_{q_\alpha}!}), \tag{4.4.19}$$

where $t_{-\alpha,m}$ this time reads

$$t_{-\alpha,m} = (q-q^{-1})e_{\delta-\alpha+m\delta}\otimes f_{\delta-\alpha+m\delta} = (q-q^{-1})q^{-1}[2]^{-1}_{q^{1/2}}K^{-1}x^-_{m+1}\otimes x^+_{-m-1}. \tag{4.4.20}$$

$R_{-\alpha}(\zeta_{12})$ becomes

$$\begin{aligned}
R_{-\alpha}(\zeta_{12}) &= \varphi^1_{\zeta_1} \otimes \varphi^1_{\zeta_2} \circ \left(1 + \sum_{m\geq 0} t_{-\alpha,m} + \sum_{m\geq 0}\sum_{n\geq 0} t_{-\alpha,m+n+1}t_{-\alpha,m} + \frac{q}{q+1}\sum_{m\geq 0} t^2_{-\alpha,m}\right)\\
&= I - \frac{\zeta_{12}\,(q^2-1)}{(\zeta_{12}-1)\,q}(\epsilon_{2,1}\otimes\epsilon_{1,2}+\epsilon_{3,2}\otimes\epsilon_{2,3}) - \frac{\zeta_{12}\,(q^2-1)}{q^2\,(\zeta_{12}+q)}\epsilon_{2,1}\otimes\epsilon_{2,3}\\
&- \frac{\zeta_{12}q\,(q^2-1)}{\zeta_{12}q+1}\epsilon_{3,2}\otimes\epsilon_{1,2} - \frac{\zeta_{12}^2(q-1)^2(q+1)\,(q^2-\zeta_{12})}{(\zeta_{12}^2-1)\,q^2\,(\zeta_{12}+q)}\epsilon_{3,1}\otimes\epsilon_{1,3}.
\end{aligned} \tag{4.4.21}$$

And the -2α term is

$$R_{-2\alpha}(\zeta_{12}) = (\varphi^1_{\zeta_1} \otimes \varphi^1_{\zeta_2} \circ \mathcal{R}_{-2\alpha}) = \varphi^1_{\zeta_1} \otimes \varphi^1_{\zeta_2} \circ \prod_{m\geq 0}(1 + t_{-2\alpha,m}), \tag{4.4.22}$$

where $t_{-2\alpha,m}$

$$\begin{aligned}
t_{-2\alpha,m} &= (q-q^{-1})e_{\delta-2\alpha+2m\delta}\otimes f_{\delta-2\alpha+2m\delta} = (q-q^{-1})[2]^{-2}_{q^{1/2}}[2]^{-1}_q q^{-4}\\
&\times K^{-2}(x^-_m x^-_{m+1} - q x^-_{m+1}x^-_m)\otimes K^2(x^+_{-m-1}x^+_{-m} - q^{-1}x^+_{-m}x^+_{-m-1}).
\end{aligned} \tag{4.4.23}$$

Performing the summations we get

$$R_{-2\alpha}(\zeta_{12}) = \varphi^1_{\zeta_1} \otimes \varphi^1_{\zeta_2} \circ \left(1 + \sum_{m\geq 0} t_{-2\alpha,m}\right) = I - \frac{\zeta_{12}\,(q^4-1)}{(\zeta_{12}^2-1)\,q^2}\epsilon_{3,1}\otimes\epsilon_{1,3}. \tag{4.4.24}$$

The last term (4.3.8) is

$$K_0 = (\varphi^1_{\zeta_1} \otimes \varphi^1_{\zeta_2} \circ \mathcal{K}) = q(\epsilon_{1,1} \otimes \epsilon_{1,1} + \epsilon_{3,3} \otimes \epsilon_{3,3}) + q^{-1}(\epsilon_{1,1} \otimes \epsilon_{3,3} + \epsilon_{3,3} \otimes \epsilon_{1,1}) + \epsilon_{1,1} \otimes \epsilon_{2,2} + \epsilon_{2,2} \otimes \epsilon_{1,1} + \epsilon_{2,2} \otimes \epsilon_{2,2} + \epsilon_{2,2} \otimes \epsilon_{3,3} + \epsilon_{3,3} \otimes \epsilon_{2,2}. \tag{4.4.25}$$

Now we gather all pieces together

$$R(\zeta_{12}) = R_{-\alpha}(\zeta_{12}) R_{-2\alpha}(\zeta_{12}) R_{\delta}(\zeta_{12}) R_{\alpha}(\zeta_{12}) R_{2\alpha}(\zeta_{12}) K_0. \tag{4.4.26}$$

We obtain the R-matrix of the Izergin–Korepin model

$$\begin{aligned} R(\zeta) &= q^{-1}(\epsilon_{1,1} \otimes \epsilon_{1,1} + \epsilon_{3,3} \otimes \epsilon_{3,3}) - \frac{(\zeta-1)}{q^2-\zeta}(\epsilon_{2,2} \otimes \epsilon_{1,1} + \epsilon_{2,2} \otimes \epsilon_{3,3}) \\ &+ \frac{(q^2-1)}{q\,(q^2-\zeta)}(\zeta\epsilon_{2,1} \otimes \epsilon_{1,2} + \zeta\epsilon_{3,2} \otimes \epsilon_{2,3} + \epsilon_{1,2} \otimes \epsilon_{2,1} + \epsilon_{2,3} \otimes \epsilon_{3,2}) \\ &- \frac{(\zeta-1)q(\zeta+q)}{(q^2-\zeta)\,(\zeta+q^3)}(\epsilon_{1,1} \otimes \epsilon_{3,3} + \epsilon_{3,3} \otimes \epsilon_{1,1}) - \frac{(\zeta-1)}{q^2-\zeta}(\epsilon_{1,1} \otimes \epsilon_{2,2} + \epsilon_{3,3} \otimes \epsilon_{2,2}) \\ &+ \frac{(-\zeta+\zeta q^5-\zeta q^4+q^4-\zeta q^3+\zeta q^2-\zeta^2 q+\zeta q)}{q\,(q^2-\zeta)\,(\zeta+q^3)}\epsilon_{2,2} \otimes \epsilon_{2,2} \\ &+ \frac{\zeta\,(q^2-1)\,(\zeta+q^3-\zeta q+q)}{q\,(q^2-\zeta)\,(\zeta+q^3)}\epsilon_{3,1} \otimes \epsilon_{1,3} + \frac{(q^2-1)\,(\zeta+q^3+\zeta q^2-q^2)}{q\,(q^2-\zeta)\,(\zeta+q^3)}\epsilon_{1,3} \otimes \epsilon_{3,1} \\ &+ \frac{(\zeta-1)\,(q^2-1)}{q\,(q^2-\zeta)\,(\zeta+q^3)}\left(-q^3\epsilon_{1,2} \otimes \epsilon_{3,2} - q^2\epsilon_{2,3} \otimes \epsilon_{2,1} + \zeta q\epsilon_{3,2} \otimes \epsilon_{1,2} + \zeta\epsilon_{2,1} \otimes \epsilon_{2,3}\right). \end{aligned} \tag{4.4.27}$$

A more conventional form of the IK R-matrix is slightly different. In order to achieve this form we need to perform a grading transformation and a similarity transformation. The appropriate similarity transformation is

$$\tilde{R}(\zeta_{12}) = S^{-1} \otimes S^{-1} R(\zeta_{12}) S \otimes S, \tag{4.4.28}$$

where

$$S = \begin{pmatrix} 0 & 0 & -iq^{1/4} \\ 0 & i & 0 \\ -iq^{1/4} & 0 & 0 \end{pmatrix}. \tag{4.4.29}$$

Next we set $\zeta_{12} = z_2^2/z_1^2$, apply the gauge transformation and multiply the result by q in order to make the first matrix element equal to 1.

$$\bar{R}(z_2/z_1) = qG(z_2) \otimes G(z_1) \tilde{R}(z_2^2/z_1^2) G(z_2)^{-1} \otimes G^{-1}(z_1). \tag{4.4.30}$$

where $G(z)$ is defined by

$$G(z) = \begin{pmatrix} z^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & z \end{pmatrix}. \tag{4.4.31}$$

The matrix denoted here by $\bar{R}$ is the same, up to a normalization, as the matrix (1.1.3) with the Boltzmann weights (1.1.4) written in the multiplicative convention

$$z_2/z_1 = e^{u/2} \quad q = -e^{-2\eta}.$$

4.5 Khoroshkin–Tolstoy formula and $V^{(1)} \otimes V^{(k)}$

In this section we will leave the second factor of the representation space of the R-matrix to be $V^{(k)}$ for any k while the first factor will be in the fundamental representation. The goal is to show how to work with $V^{(k)}$ representation concerning the KT formula. The first factor of the R-matrix in $V^{(1)} \otimes V^{(k)}$ representation allows us to truncate the exponentials of the KT formula which enables us to proceed with calculations. One could, in principle, take $V^{(k_1)} \otimes V^{(k_2)}$, however, in this case the calculations become much more complicated and it is not clear for us at the moment how to perform them. It is probably more useful to use in this situation the integral formula for the universal R-matrix [34].

In order to deal with the unspecified representation $V^{(k)}$ in the second factor of $V^{(1)} \otimes V^{(k)}$ we will substitute the Drinfeld generators $x_r^{\pm}$ and K with another set of operators. This is needed to perform the summations in the KT formula. Indeed, decomposing the operators $x_r^{\pm}$ into two parts, each of which is restricted to one direction in $V^{(k)}$, i.e. it acts either on the first or on the second index in $v_{i,j}$, we see from Eqs. (3.4.16)-(3.4.19) that the grading r of the operators $x_r^{\pm}$ can be taken out since $x_r^{\pm} \propto x_0^{\pm}$ when restricted to one of the two directions.

Introduce the algebra $\mathcal{A}_q^k$ of operators a_1, a_2, $\bar{a}_1$, $\bar{a}_2, \kappa_1, \kappa_2$, which satisfy the following commutation relations

$$[a_1,\bar{a}_2] = [a_2,\bar{a}_1] = [\kappa_1, \kappa_2] = 0, \tag{4.5.1}$$

$$\kappa_i a_j = q^{-\delta_{i,j}} a_j \kappa_i, \quad \kappa_i \bar{a}_j = q^{\delta_{i,j}} \bar{a}_j \kappa_i, \tag{4.5.2}$$

$$a_1 a_2 - a_2 a_1 \ \phi(\kappa_1,\kappa_2) = 0, \tag{4.5.3}$$

$$\bar{a}_2 \bar{a}_1 - \bar{a}_1 \bar{a}_2 \ \phi(\kappa_1,\kappa_2) = 0, \tag{4.5.4}$$

$$a_1 \bar{a}_1 = D_1(\kappa_1,\kappa_2), \quad \bar{a}_1 a_1 = D_1(q^{-1}\kappa_1,\kappa_2), \tag{4.5.5}$$

$$a_2 \bar{a}_2 = D_2(\kappa_1,\kappa_2), \quad \bar{a}_2 a_2 = D_2(\kappa_1,q^{-1}\kappa_2), \tag{4.5.6}$$

where the new functions are

$$D_1(\kappa_1,\kappa_2) = \frac{\kappa_2 q^{\frac{1}{2}-k} \left(\kappa_2^2 - \kappa_1^2\right) \left(\kappa_1^2 q^2 - 1\right) \left(\kappa_1^2 + q^{2k+1}\right)}{\kappa_1 (q-1)^2 \left(\kappa_2^2 + \kappa_1^2 q\right) \left(\kappa_1^2 + \kappa_2^2 q\right)}, \tag{4.5.7}$$

$$D_2(\kappa_1,\kappa_2) = \frac{\kappa_1 q^{\frac{1}{2}-k} \left(\kappa_2^2 q^2 - \kappa_1^2\right) \left(\kappa_2^2 q^3 + 1\right) \left(q^{2k} - \kappa_2^2\right)}{\kappa_2 (q-1)^2 \left(\kappa_1^2 + \kappa_2^2 q\right) \left(\kappa_1^2 + \kappa_2^2 q^3\right)}, \tag{4.5.8}$$

and the function ϕ is defined as

$$\phi(\kappa_1,\kappa_2) = \frac{\left(\kappa_1^2 - \kappa_2^2\right) \left(\kappa_1^2 + \kappa_2^2 q^3\right)}{\left(\kappa_2^2 + \kappa_1^2 q\right) \left(\kappa_1^2 - \kappa_2^2 q^2\right)}.$$

We call it ϕ since it has the eigenvalues on $V^{(k)}$ equal to $\phi_{i,j} = \phi_{j-i}$ with ϕ_j defined in (3.4.22).

We need to specify the normal ordering of the operators since it will be used later. The operators of $\mathcal{A}_q^k$ are normally ordered when κ_1 and κ_2 are to the right of $\bar{a}_1$, $\bar{a}_2$, a_1, a_2 and within the a's the operators with the index 1 are to the left from the operators with the index 2.

If we restrict ourselves with the representation spaces $V^{(k)}$, then we can use the algebra $\mathcal{A}_q^k$ to write the Drinfeld generators of $U_q(A_2^{(2)})$

$$K = q^k \kappa_1^{-1} \kappa_2^{-1}, \tag{4.5.9}$$
$$x_r^+ = (-1)^r q^{2r} a_1 \kappa_1^{-2r} + q^r a_2 \kappa_2^{-2r}, \tag{4.5.10}$$
$$x_r^- = q^{-r} \bar{a}_2 \kappa_2^{-2r} + (-1)^r \bar{a}_1 \kappa_1^{-2r}. \tag{4.5.11}$$

With this substitution we can perform the summations in KT formula and obtain the answer written as a 3×3 matrix with the entries in $\mathcal{A}_q^k$. Formulae (3.4.33) define the representation of $\mathcal{A}_q^k$ on $V^{(k)}$

$$\kappa_1 v_{i,j} = q^i v_{i,j}, \quad \kappa_2 v_{i,j} = q^j v_{i,j}, \tag{4.5.12}$$
$$a_1 v_{i,j} = \alpha_{i,j}^+ v_{i-1,j}, \quad a_2 v_{i,j} = \beta_{i,j}^+ v_{i,j-1}, \tag{4.5.13}$$
$$\bar{a}_1 v_{i,j} = \alpha_{i,j}^- v_{i+1,j}, \quad \bar{a}_2 v_{i,j} = \beta_{i,j}^- v_{i,j+1}. \tag{4.5.14}$$

The Drinfeld rational function $\Psi(u)$ can be written in the language of the operators of $\mathcal{A}_q^k$ using the formula (recall Eq. (3.1.6))

$$\Psi(u) = q^k \kappa_1^{-1} \kappa_2^{-1} + (q^{1/2} - q^{-1/2}) \sum_{r>0} u^r [x_r^+, x_0^-],$$

which can be written as

$$\Psi(u) = \frac{q^2 u D_1\left(\frac{\kappa_1}{q}, \kappa_2\right)}{\kappa_1^2 + q^2 u} + \frac{qu D_2\left(\kappa_1, \frac{\kappa_2}{q}\right)}{qu - \kappa_2^2} - \frac{u D_2(\kappa_1, \kappa_2)}{u - \kappa_2^2 q} - \frac{u D_1(\kappa_1, \kappa_2)}{\kappa_1^2 + u}. \tag{4.5.15}$$

Before proceeding with the computation of the $R^{(k)}$-matrix we make a remark. The operators a_i, $\bar{a}_i$, although act on one direction of the space $V^{(k)}$ (first or second index in $v_{i,j}$), have their matrix elements depending on both indices. This means that the two species $\{a_1, \bar{a}_1, \kappa_1\}$ and $\{a_2, \bar{a}_2, \kappa_2\}$ are not independent which can be seen from (4.5.3) and (4.5.4). We can pass to another algebra $\mathcal{B}_q^k$ where this does not happen. This algebra is generated by $\kappa_1, \kappa_2, b_1, b_2, \bar{b}_1, \bar{b}_2$. It has two species of operators $\{b_1, \bar{b}_1, \kappa_1\}$ and $\{b_2, \bar{b}_2, \kappa_2\}$ which are independent of each other. The algebra $\mathcal{B}_q^k$ has the following commutation relations

$$[b_1, \bar{b}_2] = [b_2, \bar{b}_1] = [b_1, b_2] = [\bar{b}_1, \bar{b}_2] = [\kappa_1, \kappa_2] = 0, \tag{4.5.16}$$
$$\kappa_i b_j = q^{-\delta_{i,j}} b_j \kappa_i, \quad \kappa_i \bar{b}_j = q^{\delta_{i,j}} \bar{b}_j \kappa_i, \tag{4.5.17}$$
$$b_1 \bar{b}_1 = \tilde{D}_1(\kappa_1), \quad \bar{b}_1 b_1 = \tilde{D}_1(q^{-1}\kappa_1), \tag{4.5.18}$$
$$b_2 \bar{b}_2 = \tilde{D}_2(\kappa_2), \quad \bar{b}_2 b_2 = \tilde{D}_2(q^{-1}\kappa_2). \tag{4.5.19}$$

The operators $\tilde{D}_{1,2}$ are defined as

$$\tilde{D}_1(\kappa_1) = -\frac{q^{\frac{3}{2}-k}\left(\kappa_1^2 q^2 - 1\right)\left(\kappa_1^2 + q^{2k+1}\right)}{\kappa_1 (q-1)^2},$$
$$\tilde{D}_2(\kappa_2) = -\frac{q^{\frac{1}{2}-k}\left(\kappa_2^2 q^3 + 1\right)\left(q^{2k} - \kappa_2^2\right)}{\kappa_2 (q-1)^2}.$$

The isomorphism between $\mathcal{A}_q^k$ and $\mathcal{B}_q^k$ is given explicitly by

$$\begin{aligned} a_1 &\mapsto \frac{\kappa_2}{\kappa_1^2 + q\kappa_2^2} b_1, \quad \bar{a}_1 \mapsto \frac{1}{\kappa_1^2 + q\kappa_2^2}\, \bar{b}_1\, (\kappa_1^2 - \kappa_2^2), \\ a_2 &\mapsto \frac{1}{\kappa_1^2 + q\kappa_2^2}\, b_2\, (\kappa_1^2 - \kappa_2^2), \quad \bar{a}_2 \mapsto \frac{\kappa_1}{\kappa_1^2 + q\kappa_2^2} \bar{b}_2. \end{aligned}$$

The algebra $\mathcal{B}_q^k$ is easier to work with and it resembles the q-oscillator algebras frequently used to study the L-matrices. Indeed, to construct the L-matrices one often looks for a q-oscillator algebra (Osc_q) which is homomorphic to a Borel subalgebra of the quantum group under consideration. It is usually done by writing an Ansatz which, after mapping Osc_q to the Borel subalgebra, must satisfies the corresponding Serre relations. This gives certain constraints which fix the Osc_q algebra up to a certain extent (see [5, 4, 81] and [15] for the $U_q(A_2^{(2)})$ algebra). In our work the q-oscillator algebra $\mathcal{B}_q^k$ was obtained constructively, i.e. directly from the representation $V^{(k)}$. We will, however, stick to the algebra $\mathcal{A}_q^k$ in what follows, as the formulae that we obtain are a bit more compact in this case.

Now we can turn to the calculation of the R-matrix. We start with the imaginary part $\mathcal{R}_\delta$. As we learned in Section 4.4 it is worth to start straight from the normalized version of this term, i.e. subtracting in the exponent the term which is obtained by acting with $h_r \otimes h_{-r}$ on the highest weight vector of the representation $V^{(1)} \otimes V^{(k)}$, denote it by v again

$$(q^{1/2} - q^{-1/2}) \frac{r\ \varphi^1_{\zeta_1}(h_r) \otimes \varphi^k_{\zeta_2}(h_{-r})}{[2r]_{q^{1/2}}(q^r + q^{-r} + (-1)^{r+1})}.v = \frac{q^{kr-r}\left(q^{kr} - q^{-kr}\right)}{r\,(q^{-r} + q^r + (-1)^{r+1})}.v. \tag{4.5.20}$$

If we write

$$\varphi^1_{\zeta_1}(h_r) \otimes \varphi^k_{\zeta_2}(h_{-r}).v = \eta_r(\zeta_{12}).v, \tag{4.5.21}$$

then we need to evaluate

$$R_\delta^{(k)} = \exp\left((q^{1/2} - q^{-1/2}) \sum_{r>0} \frac{r\ (\varphi^1_{\zeta_1}(h_r) \otimes \varphi^k_{\zeta_2}(h_{-r}) - \eta_r(\zeta_{12}))}{[2r]_{q^{1/2}}(q^r + q^{-r} + (-1)^{r+1})}\right). \tag{4.5.22}$$

Using our result (3.3.22) to write the representation for $\varphi^k_{\zeta_2}(h_{-r})$ we can evaluate the exponent. We get three diagonal terms in the exponential

$$\begin{aligned} &\frac{(-1)^{r+1} q^{-3r}\left((-1)^r q^r \kappa_2^{2r} + (-1)^{r+1} q^r + \kappa_1^{2r} - 1\right)}{r} \epsilon_{1,1} \\ &+ \frac{q^{-2r}\left(-q^{2(k+1)r} + ((-1)^r q^r - 1)\left((-1)^r q^r \kappa_2^{2r} + \kappa_1^{2r}\right) + 1\right)}{r} \epsilon_{2,2} \\ &+ \frac{-q^{2kr} + (-1)^{r+1} q^{2kr+r} + (-1)^r q^r \kappa_2^{2r} + \kappa_1^{2r}}{r} \epsilon_{3,3}. \end{aligned}$$

After performing the summation we obtain

$$R_{\delta}^{(k)}(\zeta) = \frac{(\zeta\kappa_1^2 + q^3)(q^2 - \zeta\kappa_2^2)}{(q^2 - \zeta)(\zeta + q^3)}\epsilon_{1,1} + \frac{\left(1 - \zeta q^{2k}\right)(q^2 - \zeta\kappa_1^2)(\zeta\kappa_2^2 + q)}{(1 - \zeta\kappa_2^2)(q^2 - \zeta)(\zeta\kappa_1^2 + q)}\epsilon_{2,2}$$
$$+ \frac{\left(1 - \zeta q^{2k}\right)\left(\zeta q^{2k+1} + 1\right)}{(1 - \zeta\kappa_1^2)(\zeta q\kappa_2^2 + 1)}\epsilon_{3,3}. \tag{4.5.23}$$

Now we derive the other terms of the R-matrix. The α term reads

$$R_{\alpha}^{(k)}(\zeta) = I + \epsilon_{1,2}\left(\bar{a}_1\frac{(q^2-1)}{[2]_{q^{1}/2}^{1/2}(\zeta\kappa_1^2 + q)} - \bar{a}_2\frac{(q-1)[2]_{q^{1}/2}^{1/2}}{\sqrt{q}(\zeta\kappa_2^2 - 1)}\right)$$
$$+ \epsilon_{2,3}\left(\bar{a}_2\frac{(q-1)[2]_{q^{1}/2}^{1/2}}{\sqrt{q}(\zeta\kappa_2^2 q + 1)} - \bar{a}_1\frac{(q-1)[2]_{q^{1}/2}^{1/2}}{\sqrt{q}(\zeta\kappa_1^2 - 1)}\right) +$$
$$+ \epsilon_{1,3}\Bigg(\bar{a}_2\bar{a}_1\frac{(q-1)^2(\zeta\kappa_2^2 + q)}{q^{3/2}(\zeta\kappa_2^2 - 1)(\zeta^2\kappa_1^2\kappa_2^2 - 1)} - \bar{a}_1\bar{a}_2\frac{(q-1)^2(q^2 - \zeta\kappa_1^2)}{q^{3/2}(\zeta^2\kappa_1^2\kappa_2^2 - 1)(\zeta\kappa_1^2 + q)}$$
$$- \bar{a}_2^2\frac{(q-1)^2(\zeta\kappa_2^2 q + 1)}{\sqrt{q}(\zeta\kappa_2^2 q^2 - 1)(\zeta^2\kappa_2^4 q^3 + 1)} - \bar{a}_1^2\frac{(q-1)^2(\zeta\kappa_1^2 - 1)}{\sqrt{q}(\zeta\kappa_1^2 q + 1)(\zeta^2\kappa_1^4 q + 1)}\Bigg), \tag{4.5.24}$$

the 2α is

$$R_{2\alpha}^{(k)}(\zeta) = I + \epsilon_{1,3}\Bigg(- \bar{a}_2^2\frac{\zeta\kappa_2^2(q-1)^2\sqrt{q}}{\zeta^2\kappa_2^4 q^3 + 1} + \bar{a}_1^2\frac{\zeta\kappa_1^2(q-1)^2}{\sqrt{q}(\zeta^2\kappa_1^4 q + 1)}$$
$$- \bar{a}_1\bar{a}_2\frac{\zeta\kappa_1^2\kappa_2^2(q-1)^2(q+1)(q^2+1)(\kappa_1^2 + \kappa_2^2 q)}{q^{3/2}(\zeta^2\kappa_1^2\kappa_2^2 - 1)(\kappa_2^2 + \kappa_1^2 q)(\kappa_2^2 q^2 - \kappa_1^2)}\Bigg). \tag{4.5.25}$$

The $-\alpha$ term

$$R_{-\alpha}^{(k)}(\zeta) = I + \epsilon_{2,1}\left(a_2\frac{\zeta\kappa_2(q-1)[2]_{q^{1}/2}^{1/2}q^{k-\frac{1}{2}}}{\kappa_1(q^2 - \zeta\kappa_2^2)} - a_1\frac{\zeta\kappa_1(q-1)[2]_{q^{1}/2}^{1/2}q^{k-\frac{1}{2}}}{\kappa_2(\zeta\kappa_1^2 + q^3)}\right)$$
$$+ \epsilon_{3,2}\left(a_1\frac{\zeta\kappa_1(q-1)[2]_{q^{1}/2}^{1/2}q^{k+\frac{1}{2}}}{\kappa_2(q^2 - \zeta\kappa_1^2)} - a_2\frac{\zeta\kappa_2(q-1)[2]_{q^{1}/2}^{1/2}q^{k+\frac{1}{2}}}{\kappa_1(\zeta\kappa_2^2 + q)}\right) +$$
$$+ \epsilon_{3,1}\Bigg(a_2 a_1\frac{\zeta^2(q-1)^2 q^{2k+\frac{1}{2}}(q^4 - \zeta\kappa_1^2)}{(\zeta\kappa_1^2 + q^3)(q^4 - \zeta^2\kappa_1^2\kappa_2^2)} - a_1^2\frac{\zeta^2\kappa_1^2(q-1)^2 q^{2k+\frac{1}{2}}(q^4 - \zeta\kappa_1^2)}{\kappa_2^2(\zeta\kappa_1^2 + q^3)(\zeta^2\kappa_1^4 + q^7)}$$
$$- a_2^2\frac{\zeta^2\kappa_2^2(q-1)^2 q^{2k+\frac{1}{2}}(\zeta\kappa_2^2 + q^3)}{\kappa_1^2(q^2 - \zeta\kappa_2^2)(\zeta^2\kappa_2^4 + q^5)} + a_1 a_2\frac{\zeta^2(q-1)^2 q^{2k+\frac{1}{2}}(\zeta\kappa_2^2 + q^3)}{(q^2 - \zeta\kappa_2^2)(q^4 - \zeta^2\kappa_1^2\kappa_2^2)}\Bigg), \tag{4.5.26}$$

and finally the -2α term

$$R_{-2\alpha}^{(k)}(\zeta) = I + \epsilon_{3,1}\Bigg(a_1^2\frac{\zeta(q-1)^2 q^{2k+\frac{9}{2}}}{\kappa_2^2(\zeta^2\kappa_1^4 + q^7)} - a_2^2\frac{\zeta(q-1)^2 q^{2k+\frac{7}{2}}}{\kappa_1^2(\zeta^2\kappa_2^4 + q^5)}$$
$$- a_1 a_2\frac{\zeta(q-1)^2(q+1)(q^2+1)q^{2k+\frac{5}{2}}(\kappa_1^2 + \kappa_2^2 q)}{(\kappa_1^2 - \kappa_2^2)(\kappa_1^2 + \kappa_2^2 q^3)(q^4 - \zeta^2\kappa_1^2\kappa_2^2)}\Bigg). \tag{4.5.27}$$

In these formulae we performed the commutations between a_1 and a_2 where it lead to more compact expressions.

Next we need to multiply all the terms together according to the ordering in (4.3.1). We write the result in the following form

$$R^{(k)}(\zeta) = g(\zeta) \begin{pmatrix} \mathcal{A}_1 & \mathcal{B}_1 & \mathcal{B}_2 \\ \mathcal{C}_1 & \mathcal{A}_2 & \mathcal{B}_3 \\ \mathcal{C}_2 & \mathcal{C}_3 & \mathcal{A}_3 \end{pmatrix}, \tag{4.5.28}$$

where we have

$$g(\zeta) = \frac{1}{(q^2-\zeta)(\zeta+q^3)},$$

$$\begin{aligned}
&\mathcal{A}_1 = \lambda_1^{(1)}, \quad \mathcal{A}_2 = \lambda_2^{(1)} + \lambda_2^{(2)}\bar{a}_1 a_2 + \lambda_2^{(3)}\bar{a}_2 a_1, \\
&\mathcal{A}_3 = \lambda_3^{(1)} + \lambda_3^{(2)}\bar{a}_1 a_2 + \lambda_3^{(3)}\bar{a}_2 a_1 + \lambda_3^{(4)}\bar{a}_1^2 a_2^2 + \lambda_3^{(5)}\bar{a}_2^2 a_1^2, \\
&\mathcal{B}_1 = \nu_1^{(1)}\bar{a}_1 + \nu_1^{(2)}\bar{a}_2, \quad \mathcal{B}_2 = \nu_2^{(1)}\bar{a}_1^2 + \nu_2^{(2)}\bar{a}_2^2 + \nu_2^{(3)}\bar{a}_1\bar{a}_2, \\
&\mathcal{B}_3 = \nu_3^{(1)}\bar{a}_1 + \nu_3^{(2)}\bar{a}_2 + \nu_3^{(3)}\bar{a}_1^2 a_2 + \nu_3^{(4)}\bar{a}_2^2 a_1, \\
&\mathcal{C}_1 = \mu_1^{(1)} a_1 + \mu_1^{(2)} a_2, \quad \mathcal{C}_2 = \mu_2^{(1)} a_1^2 + \mu_2^{(2)} a_2^2 + \mu_2^{(3)} a_1 a_2, \\
&\mathcal{C}_3 = \mu_3^{(1)} a_1 + \mu_3^{(2)} a_2 + \mu_3^{(3)}\bar{a}_2 a_1^2 + \mu_3^{(4)}\bar{a}_1 a_2^2.
\end{aligned} \tag{4.5.29}$$

Coefficients λ, μ and ν are written in Appendix A. The $R^{(k)}$-matrix satisfies the equation

$$PR(x/y)R^{(k)}(x)R^{(k)}(y) = R^{(k)}(y)R^{(k)}(x)PR(x/y). \tag{4.5.30}$$

Taking the products of appropriately normalised $R^{(k)}$-matrices and performing the trace we obtain the transfer matrices $T_k(z) = T_{V^{(k)}(z)}$. These transfer matrices commute among each other because of the Yang–Baxter equation. They also satisfy quadratic equations called the TT-relations. These relations can be found using the q-character theory [60, 61], see also [86, 87, 88]. Below we write the TT-relation [88] satisfied by the transfer matrices constructed from $R^{(k)}$

$$T_{k+1}(\zeta)T_{k-1}\left(\zeta q^{-1}\right) = T_k(\zeta)T_k\left(\zeta q^{-1}\right) + q^k \prod_{i=1}^{n} \frac{z_i^2 q^k - \zeta^2 q^{-k}}{z_i^2 - \zeta^2} T_k(iq^{-1/2}\zeta), \tag{4.5.31}$$

where we denoted by n the length of the physical space, since the previously used letter L in this chapter is used for other purposes. In order to check Eq. (4.5.31) one first needs to perform the transformation as in (4.4.30). One must take the $\bar{R}^{(k)}$-matrix

$$\bar{R}^{(k)}(\zeta) = \zeta^{-4}G(\zeta^{-1})S^{-1}R^{(k)}(\zeta^{-2})SG^{-1}(\zeta^{-1}), \tag{4.5.32}$$

with G and S defined previously (4.4.31) and (4.4.29).

4.6 L-operators

Along with the transfer matrices constructed with higher dimensional representations $V^{(k)}$ in the auxiliary space there is another important set of transfer matrices. These transfer matrices are obtained from the monodromy matrices with infinite dimensional auxiliary spaces. In the untwisted algebras one often uses oscillator representations [5, 4, 81, 16, 14, 15, 17] in the auxiliary space. Transfer matrices constructed in this way we call the representation theoretic Q-operators, or simply the Q-operators. We define these operators as

$$Q_V(z) = Tr_{V^{(\infty)}} \kappa \mathcal{M}_{V^{(\infty)}}(z), \tag{4.6.1}$$

where κ is some element of the Cartan subalgerba and $V^{(\infty)}$ is an infinite dimensional representation of the Borel subalgebra $U_q(\mathfrak{b}^-)$ which cannot be extended to a representation of the full quantum group. Normally one expects that the monodromy matrix $\mathcal{M}_{V^{(\infty)}}(z)$ in Eq. (4.6.1) can be written as a product of L-operators, where L can be defined as a representation of the universal $\mathcal{R}$-matrix on $V^{(1)} \otimes V^{(\infty)}$. Therefore the first task is to find representations $V^{(\infty)}$ and then to compute the corresponding L-operators. Using these L-operators one can show that the corresponding Q-operators commute with the transfer matrix

$$[Q(\xi),T(\zeta)] = 0, \tag{4.6.2}$$

if κ in (4.6.1) agrees with the one in (3.2.4).

In the original works of Baxter the Q-operators were defined as operators whose eigenvalues are polynomials in the spectral parameter with the roots equal to the Bethe roots [3, 2]. To these Q-operators we assign the index B. Understanding relations between the representation theoretic Q-operators and the Baxter's Q^B-operators is, in general, a hard task. Such relations are well studied for the algebras $U_q(\hat{sl}_2)$ and $U_q(\hat{sl}_3)$ in [5, 4].

Taking into account that the roots of the operators Q^B are the Bethe roots [2] the equation for the eigenvalue of the transfer matrix (1.2.17) can be rewritten as

$$\begin{aligned} q^{-n}T(\zeta)Q^B(\zeta)Q^B(\sqrt{-q}\zeta) &= \frac{F(\zeta)}{F(q\zeta)}Q^B(\sqrt{-q^{-1}}\zeta)Q^B(q\zeta) \\ &+ \frac{F(\zeta)F(\sqrt{-q}\zeta)}{qF(q\zeta)F(\sqrt{-q^3}\zeta)}Q^B(\zeta)Q^B(\sqrt{-q^3}\zeta) + qQ^B(q^{-1}\zeta)Q^B(\sqrt{-q}\zeta), \end{aligned} \tag{4.6.3}$$

where

$$F(\zeta) = \prod_{i=1}^{n} \left(\zeta^2 - z_i^2\right).$$

This equation is called the TQ-equation corresponding to the Tarasov's Bethe ansatz for the IK model. The main motivation for our study of the Q-operators was

2. In fact, the operator Q^B in this case has the eigenvalues equal to $\prod_i(\zeta^2 - \zeta_i^2)$, where ζ_i are Bethe roots. In the usual case it would be $\prod_i(\zeta - \zeta_i)$.

to construct the Q-operator that satisfies (4.6.3) using the representation theory approach. At the moment this problem remains open. Below we propose a way of computing L-operators based on the representations obtained in Chapter 3. The first Q-operator that we compute essentially coincides with the one constructed[3] in [15].

If we take a limit $k \to \infty$ in $V^{(1)} \otimes V^{(k)}$ and evaluate the universal $\mathcal{R}$-matrix on the resulting space, we will obtain a L-matrix. One can take the limit of the representation $V^{(k)}$ in different ways which correspond to different Drinfeld's fractions, thus leading to different L-matrices.

The easiest way to take the limit $k \to \infty$ is to redefine beforehand the reference vector of the space represented in Fig. 3.3. Choosing the vector $v_{k,k}$ to be the reference vector we see that the limit $k \to \infty$ sends to infinity symmetrically both wings of the representation space Fig. 3.3. This redefinition of the reference vector amounts to rescaling the operator κ_2 and leaving untouched κ_1. We write

$$\kappa_1 \to \lambda_1, \quad \kappa_2 \to q^k \lambda_2^{-1}, \tag{4.6.4}$$

and assuming the u-basis: $v_{i,j} = u_{i,k-j}$ we get

$$\lambda_1 u_{i,j} = q^i u_{i,j}, \quad \lambda_2 u_{i,j} \to q^j u_{i,j}.$$

The module obtained in such a way corresponds to the Drinfeld fraction

$$\Psi^{(0)}(u) = \frac{(1-qu)}{(1+u)}. \tag{4.6.5}$$

Note, in the u-basis there is no restriction on the indices $i \leq j$ as for $v_{i,j}$. The only restriction is that $i + j = k$, which is symmetric in i and j. Upon taking the limit $k \to \infty$, assuming $|q| < 1$, the algebra becomes $\mathcal{A}^{(\infty)}$ and the relations of its generators (4.5.1)-(4.5.6) must be modified. In order to make the formulae compact it is also useful to rescale the generators a_i and $\bar{a}_i$ by a factor of $[2]_{q^{1/2}}^{1/2}(q^2-1)^{-1}$. We obtain the algebra $\mathcal{A}^{(\infty)}$ generated by c_1, $c_1^\dagger$, c_2, $c_2^\dagger$ and λ_1, λ_2, where c_i and $c_i^\dagger$ replace a_i and $\bar{a}_i$, respectively, while λ_1 and λ_2 are defined in (4.6.4). Generators of $\mathcal{A}^{(\infty)}$ satisfy

$$[c_1,c_2^\dagger] = [c_2,c_1^\dagger] = [\lambda_1,\lambda_2] = 0, \tag{4.6.6}$$

$$\lambda_1 c_1 = q^{-1} c_1 \lambda_1, \quad \lambda_1 c_1^\dagger = q c_1^\dagger \lambda_1, \tag{4.6.7}$$

$$\lambda_2 c_2 = q c_2 \lambda_2, \quad \lambda_2 c_2^\dagger = q^{-1} c_2^\dagger \lambda_2, \tag{4.6.8}$$

$$\lambda_1 c_2 = c_2 \lambda_1, \quad \lambda_1 c_2^\dagger = c_2^\dagger \lambda_1, \tag{4.6.9}$$

$$\lambda_2 c_1 = c_1 \lambda_2, \quad \lambda_2 c_1^\dagger = c_1^\dagger \lambda_2, \tag{4.6.10}$$

$$c_1 c_2 - c_2 c_1\, q^{-1} = 0, \tag{4.6.11}$$

$$c_2^\dagger c_1^\dagger - c_1^\dagger c_2^\dagger\, q^{-1} = 0, \tag{4.6.12}$$

$$c_1 c_1^\dagger = \frac{(q+1)\,(1-\lambda_1^2 q^2)}{\lambda_1 \lambda_2}, \quad c_1^\dagger c_1 = -\frac{(\lambda_1^2-1)\,q(q+1)}{\lambda_1\lambda_2}, \tag{4.6.13}$$

$$c_2 c_2^\dagger = -\frac{(\lambda_2^2-1)\,q(q+1)}{\lambda_1\lambda_2}, \quad c_2^\dagger c_2 = \frac{(q+1)\,(1-\lambda_2^2 q^2)}{\lambda_1\lambda_2}. \tag{4.6.14}$$

3. In the paper [15] the L-matrix was constructed from which the Q-operator can be obtained using (4.6.1).

One could, of course, apply a transformation to c_i and $c_i^\dagger$ in such a way that the operators with the index 1 commute with those with the index 2.

Define the L-matrix as

$$R^{(\infty)}(\zeta) = \frac{1}{q^3\left(q^2-\zeta\right)\left(\zeta+q^3\right)} L(\zeta). \tag{4.6.15}$$

Take the $R^{(k)}$-matrix (4.5.28) and the coefficients from Appendix A, set there (4.6.4) and send $k \to \infty$. The resulting matrix $L(\zeta)$ is equal to

$$\begin{pmatrix} \frac{\lambda_2 q^8}{\lambda_1} + \zeta\lambda_1\lambda_2 q^5 & q^7 c_1^\dagger + q^4 c_2^\dagger\left(\zeta\lambda_1^2+q^3\right) & c_1^{\dagger 2}\frac{\lambda_1 q^7}{\lambda_2(q+1)} + c_1^\dagger c_2^\dagger \frac{\lambda_1 q^6}{\lambda_2} + c_2^{\dagger 2}\frac{\lambda_1 q^4\left(\zeta\lambda_1^2+q^3\right)}{\lambda_2(q+1)} \\ -q^4 c_1\zeta\lambda_2^2 & q^4(q^4-\zeta) - q^2 c_2^\dagger c_1\zeta\lambda_1\lambda_2 & c_1^\dagger\frac{\lambda_1 q^7}{\lambda_2} + c_2^\dagger\frac{\lambda_1 q^3\left(q^4-\zeta\right)}{\lambda_2} - c_2^{\dagger 2} c_1\frac{\zeta\lambda_1^2 q}{q+1} \\ c_1^2\frac{\zeta\lambda_2^3 q^5}{\lambda_1(q+1)} & c_2^\dagger c_1^2\frac{\zeta\lambda_2^2 q^2}{q+1} + c_1\frac{\zeta\lambda_2 q^4}{\lambda_1} & q^2 c_2^\dagger c_1\zeta + c_2^{\dagger 2} c_1^2\frac{\zeta\lambda_1\lambda_2}{(q+1)^2} + \frac{q^3\left(\zeta+\lambda_1^2 q^5\right)}{\lambda_1\lambda_2} \end{pmatrix}. \tag{4.6.16}$$

This L operator satisfies the RLL-relation that reads

$$PR(x/y)L(x)\otimes L(y) = L(y)\otimes L(x)PR(x/y). \tag{4.6.17}$$

Using this L operator we can construct a Q-operator using the formula (4.6.1). This Q-operator must satisfy certain functional relations. In order to find these functional relations one must understand other limits $k \to \infty$ of the representation space $V^{(k)}$. These limits will lead to new L-operators which must lead to new Q-operators. The construction of the latter Q-operators from the L-matrices, however, is a hard problem which remains unsolved.

In the conclusion of this chapter let us discuss another limit $k \to \infty$ of $V^{(k)}$. Recall the asymptotic representation discussed in Section 3.5 of Chapter 3. This asymptotic representation corresponds to the Drinfeld fraction

$$\tilde{\Psi}_0 = \frac{1}{1-q^{-1}u}.$$

The asymptotic representations are defined in [62] using the the Drinfeld–Jimbo generators. We prefer to use the Drinfeld generators or, more precisely, the operators of the algebra $\mathcal{A}_q^k$. In the language of the operators of the algebra $\mathcal{A}_q^k$, in order to regularize the representation for the purpose of sending k to infinity, we simply need to rescale the generators a_1 and a_2 by multiplying them with the factor q^{-k}. This must be done on an earlier stage of the KT calculation, i.e. before we use any commutation relations. After that, we can compute the corresponding L-matrix. Unfortunately, the final expression is too large and we already have the $R^{(k)}$-matrix which takes three pages in Appendix A. The large expression for this L-matrix is not a big problem, we could still do the calculations. However, because of the terms of the form

$$\left(q^x\kappa_2^2+\kappa_1^2\right)\left(q^y\kappa_1^2+\kappa_2^2\right),$$

(where x and y are some integers) which appear in the denominators of some matrix elements due to Eqs. (4.5.5) and (4.5.6), it becomes very complicated to take the trace.

At the moment we are not aware of a solution to this problem. In the previous case, when we were working with the representation associated with the Drinfeld rational fraction (4.6.5), this problem was absent since the rescaling of κ_2 in (4.6.4) and the limit $k \to \infty$ killed the unwanted denominators in Eq. (4.5.5) and Eq. (4.5.6) leading to the formulae (4.6.13) and (4.6.14) like in the usual q-boson algebras.

Chapter 5

The twisted ground state at $q^3 = -1$

In Chapter 1 we discussed an approach to the diagonalization of the transfer matrix of the IK model. We showed how to construct the eigenstates of the transfer matrix and presented a formula for the eigenvalues. The resulting formulae for the eigenstates and eigenvalues, however, depend on the Bethe roots. The Bethe roots satisfy the Bethe Ansatz equations which are given in (1.2.18). If we want to obtain the solution, we need to solve the Bethe Ansatz equations. In general, further finite size exact analytical calculations are not possible. However, there are few instances when the Bethe equations simplify. This happens, in particular, when $q^3 = -1$ and $q^4 = -1$.

A similar phenomena appears in the XXZ-spin chain. The specification to $q^3 = 1$, called the combinatorial point, leads to a simplified ground state [113, 106, 1, 107]. In order to compute the entries of the ground state an alternative to BA technique based on certain difference equations was developed [49, 108]. These difference equations are sometimes referred to as the quantum Knizhnik–Zamolodchikov (qKZ) equations. The problem can also be reformulated in the loop basis, which corresponds to passing to the dense O($n = 1$) Temperley–Lieb model [49, 127, 47, 51, 44, 130]. This approach led to the computation of the ground state elements, their sum rules and some expectation values [49, 129, 52, 33, 18, 63]. It is possible to write the elements of the ground state in the form of a contour integral [69, 51, 108] which comes from certain expectation values of the vertex operators of the quantum group $U_q(\hat{sl}_2)$. This approach proved to be very powerful also for the combinatorial points of higher spin representations [50, 128, 45]. The method of difference equations (qKZ) is possible to extend to the case of the Izergin–Korepin model at $q^3 = -1$ (for the loop version see [46, 54, 57, 58]) and $q^4 = -1$. In the case of $q^4 = -1$ one can obtain integral formulae for the components of the ground state. They can be computed using the vertex operators for $U_q(A_2^{(2)})$ which were found in [70]. We choose, however, to follow the ABA approach.

In this chapter we will consider the (conjectured) ground state of the transfer matrix at a root of unity $q^3 = -1$ with the twist $\kappa = q$, where κ in this chapter is understood in the sense of Eq. (1.1.9). We observed for small systems that the state we are dealing with is indeed the ground state. The proof of this claim is absent at this stage. In the text below we will, nevertheless, refer to this state as to the ground state.

The ground state eigenvalue of the appropriately normalized transfer matrix become a simple polynomial in the inhomogeneities, as we will see. This means that the entries of the normalized ground state should be also polynomials in the inhomogeneities. In Section 5.1 we obtain the Q-function and show that it satisfies the BAE. Next we find the T-matrix eigenvalue and using the TT-system (4.5.31) we find eigenvalues for some higher spin transfer matrices. In Section 5.2 we use our knowledge of the Q-function to find the small size exact ground state components written as polynomials in the inhomogeneities. Also, we write the expression for the scalar product for $N = 2$. The connection of the IK model with the Temperley–Lieb dilute $\mathrm{O}(n)$ is discussed in Appendix C. In particular, using this connection the ground state components of the Temperley–Lieb dilute O(1) loop model appearing in [48, 54] can be matched with those from Section 5.2 here.

5.1 Bethe equations at $q^3 = -1$

The ground state of the IK model on a chain of length $L = N$ corresponds to a state with N particles (1.2.6). In this section we find the solution of the BAE (1.2.18) for the ground state of the IK model at $q^3 = -1$ with the twist $\kappa = q$ (in the sense of (1.1.9)). In this specific case the equations simplify and we are able to find the Q-function. The Q-function $Q(\zeta)$ is the generating function of the elementary symmetric polynomials of the Bethe roots. Since all quantities (entries of the eigenstates and the transfer matrix eigenvalues) are symmetric in the Bethe roots it is possible to express them in the basis of the elementary symmetric polynomials in the Bethe roots. This gives the explicit answer. For small systems this allows us to write the entries of the grounds state vector. Let us introduce once more the R-matrix which has the same form as (1.1.3) but this time the weights are reduced due to the fact that we set $q^3 = -1$. We will replace the parameter q by $\omega = \mathrm{e}^{i\pi/3}$. The R-matrix weights become

$$
\begin{aligned}
&x_1(\zeta) = \frac{\zeta\omega + 1}{\zeta + \omega}, \quad x_2(\zeta) = \frac{(\zeta^2 - 1)\,\omega}{(\zeta + \omega)(\zeta\omega - 1)},\\
&x_3(\zeta) = \frac{\zeta - \omega}{\zeta\omega - 1}, \quad x_4(\zeta) = \frac{(\zeta^2 - 1)\,\omega}{(\zeta + \omega)(\zeta\omega - 1)},\\
&x_5(\zeta) = \frac{\zeta(\omega - 2)}{(\zeta + \omega)(\zeta\omega - 1)}, \quad x_6(\zeta) = \frac{\zeta(\omega - 2)\sqrt{\omega}}{(\zeta + \omega)(\zeta\omega - 1)},\\
&x_7(\zeta) = \frac{\zeta^2(2\omega - 1)}{(\zeta + \omega)(\zeta\omega - 1)}, \quad y_5(\zeta) = \frac{\zeta(\omega - 2)}{(\zeta + \omega)(\zeta\omega - 1)},\\
&y_6(\zeta) = \frac{\zeta\sqrt{\omega}(2\omega - 1)}{(\zeta + \omega)(\zeta\omega - 1)}, \quad y_7(\zeta) = -\frac{\omega + 1}{(\zeta + \omega)(\zeta\omega - 1)}.
\end{aligned} \tag{5.1.1}
$$

These Boltzmann weights are written in the multiplicative convention, i.e. one must take the weights of (1.1.4) and switch there to ζ and q using $\zeta = e^{u/2}$, $q = -e^{-2\eta}$. Replacing further q with $\omega = \mathrm{e}^{i\pi/3}$ one notices that all weights (1.1.4) have a common factor of $\zeta^2 - 1$. Dividing the result by

$$
-\frac{2\zeta^2}{(\zeta^2 - 1)\sqrt{\omega}(\zeta + \omega)(\zeta\omega - 1)},
$$

we obtain (5.1.1). In this chapter we will only use the R-matrix with the Boltzmann weights (5.1.1). Let us define the action of our R-matrix on the space $\mathcal{H} = V_{z_1} \otimes \cdots \otimes V_{z_L}$, where the index L in $\mathcal{H}$ is implied. The matrix $R_{i,i+1}(z_i,z_{i+1}) = R_{i,i+1}(z_{i+1}/z_i)$ acts non trivially only on the part $V_{z_i} \otimes V_{z_{i+1}}$ of the space $\mathcal{H}$. Define the spin basis in the space $\mathcal{H}$ as follows. Let R act as

$$R(z) v_{j_1} \otimes v_{j_2} = \sum_{k_1,k_2=\pm,0} v_{k_1} \otimes v_{k_2} R(z)^{j_1,j_2}_{k_1,k_2}, \tag{5.1.2}$$

where the ordering of the basis in $V \otimes V$ is $v_+ \otimes v_+$, $v_+ \otimes v_0$, $v_+ \otimes v_-$, $v_0 \otimes v_+$, $v_0 \otimes v_0$, $v_0 \otimes v_-$, $v_- \otimes v_+$, $v_- \otimes v_0$, $v_- \otimes v_-$. We also use the spin notation, i.e. the labels $\{\uparrow, 0, \downarrow\}$ for $\{+,0,-\}$

$$\begin{aligned} v_+ &= v_\uparrow = |\uparrow\rangle = (1,0,0), \quad v_0 = v_0 = |0\rangle = (0,1,0), \\ v_- &= v_\downarrow = |\downarrow\rangle = (0,0,1). \end{aligned} \tag{5.1.3}$$

The space $\mathcal{H}$ splits into sectors of fixed spin, i.e. into subspaces $\mathcal{H}_s$ which are spanned by the vectors which have the total number of up spins $\uparrow$ minus the total number of down spins $\downarrow$ equal to the integer s. This is the same splitting of $\mathcal{H}$ as considered in Chapter 1. Our ground state is antiferromagnetic, this means that the total spin s must be minimal $s = 0$, or assuming that $\downarrow + \uparrow = 0$, the sum of all spins must add up to 0. Thus we restrict in what follows to the subspace $\mathcal{H}_0$.

In Chapter 1 the N-particle eigenstates were denoted by Ψ_N. The ground state with N particles in a system of length N will be denoted by $\Psi_N^{(0)}$. The vector $\Psi_N^{(0)}$ in $\mathcal{H}_0$ depends on the set of Bethe roots $\zeta_1,..,\zeta_N$, which we need to find, and on the inhomogeneity parameters $z_1,..,z_N$. The eigenstates are symmetric in the Bethe roots up to an overall factor which can be derived from the formula (1.2.5). If we write

$$\bar{\Psi}_N(\zeta_1,..,\zeta_N) = \prod_{i<j}^{N} \left(\omega^2 \zeta_j^2 - \zeta_i^2\right) \Psi_N(\zeta_1,..,\zeta_N), \tag{5.1.4}$$

then $\bar{\Psi}_N$ is symmetric in the Bethe roots. Solving the BAE means finding the functions $\zeta_1(z_1,..,z_N),..,\zeta_N(z_1,..,z_N)$. In fact, we find the elementary symmetric polynomials, defined in (2.5.2), of the Bethe roots $E_i(\zeta_1,..,\zeta_N)$. After that we express the ground state $\bar{\Psi}_N^{(0)}$ in the basis of the elementary symmetric polynomials

$$\tilde{\Psi}_N^{(0)}(E_1(\zeta_1,..,\zeta_N),..,E_N(\zeta_1,..,\zeta_N); z_1,..,z_N).$$

We substitute the E's and find the ground state vector which depends only on the inhomogeneities. Let us, by the abuse of notation, use the letter Ψ again for the final result, so $\bar{\Psi}_N^{(0)}(\zeta_1,..,\zeta_N; z_1,..,z_N) = \Psi_N(z_1,..,z_N)$. The vector Ψ_N has the entries denoted by $\psi_{n_1,..,n_N}(z_1,..,z_N)$

$$\Psi_N(z_1,..,z_N) = \sum_{\substack{n_1,..,n_L=\uparrow,0,\downarrow \\ n_1+..+n_N=0}} \psi_{n_1,..,n_N}(z_1,..,z_N) |n_1,..,n_N\rangle. \tag{5.1.5}$$

In order to find the components $\psi_{n_1,...,n_N}(z_1,..,z_N)$ we need to obtain the symmetric polynomials $E_N(\zeta_1,..,\zeta_N)$. Let us now turn to their calculation.

The BAE (1.2.18) in the multiplicative convention with the twist $\kappa = q$ read

$$\prod_{i=1}^{N} \frac{\left(z_i^2 - q^2\zeta_j^2\right)}{q\left(z_i^2 - \zeta_j^2\right)} - q^{-1} \prod_{\substack{i=1 \\ i\neq j}}^{N} \frac{\left(\zeta_i^2 - q^2\zeta_j^2\right)\left(q\zeta_i^2 + \zeta_j^2\right)}{\left(q^2\zeta_i^2 - \zeta_j^2\right)\left(\zeta_i^2 + q\zeta_j^2\right)} = 0, \tag{5.1.6}$$

which must hold for all ζ_j, $j = 1,..,N$. It can be easily verified that one of ζ_j must be equal to zero as a consequence of the choice of the twist. Hence, $Q(\zeta)$ is proportional to ζ^2. Notice that Eq. (5.1.6) depends on the squares of the Bethe roots. This means if the numbers $\zeta_1,..,\zeta_N$ solve the BAE, then changing the sign of any ζ_i will also lead to a solution of BAE. Because of this we introduce the new roots $\lambda_i = \zeta_i^2$. Recall that the eigenvalue depends on a spectral parameter which we denoted by ζ in (1.2.17), hence we also introduce the parameter $\lambda = \zeta^2$.

It is convenient to rewrite the Bethe equations using the functions

$$Q(\zeta(\lambda)) = Q'(\lambda) = \prod_{i=1}^{N} (\lambda - \lambda_i), \qquad F(\zeta(\lambda)) = F'(\lambda) = \prod_{i=1}^{N} \left(\lambda - z_i^2\right). \tag{5.1.7}$$

After expressing (5.1.6) in terms of Q' and F' we get

$$\frac{F'\left(q^2\lambda_j\right)}{F'\left(\lambda_j\right)} + \frac{Q'\left(-q^{-1}\lambda_j\right)Q'\left(q^2\lambda_j\right)}{qQ'\left(q^{-2}\lambda_j\right)Q'\left(-q\lambda_j\right)} = 0. \tag{5.1.8}$$

When q is a third root of minus one $q = \omega$ the Bethe equations simplify

$$\frac{F'\left(\omega^2\lambda_j\right)}{F'\left(\lambda_j\right)} + \frac{Q'\left(\omega^2\lambda_j\right)^2}{\omega Q'\left(\omega^{-2}\lambda_j\right)^2} = 0. \tag{5.1.9}$$

We can also rewrite it as

$$\omega F'\left(\omega^2\lambda_j\right)Q'\left(\omega^{-2}\lambda_j\right)^2 + F'\left(\lambda_j\right)Q'\left(\omega^2\lambda_j\right)^2 = 0. \tag{5.1.10}$$

If we substitute λ_j with a parameter t in the right hand side of (5.1.10), then the expression

$$G(t) = \omega F'\left(t\omega^2\right)Q'\left(-t\omega\right)^2 + F'\left(t\right)Q'\left(t\omega^2\right)^2, \tag{5.1.11}$$

must vanish whenever $t = \lambda_j$ for any $j = 1,..,N$. This means that $G(t)$ is proportional to $Q'(t)$. This can also be seen by looking at the transfer matrix eigenvalue. The ground state eigenvalue of the transfer matrix with the twist $\kappa = q$, $q = \omega$, written in terms of the λ roots becomes

$$\Lambda'(t) = \frac{\omega F'(t)Q'\left(t\omega^2\right)^2}{Q'(t)F'\left(t\omega^2\right)Q'(-t\omega)} + \frac{Q'(t)F'(-t\omega)}{\omega F'\left(t\omega^2\right)Q'(-t\omega)} + \frac{\omega Q'(-t\omega)}{Q'(t)}, \tag{5.1.12}$$

where we put $\Lambda'(t) = \Lambda(\sqrt{t})$. Rearranging this we find

$$\begin{aligned} &\Lambda'(t)F'\left(t\omega^2\right)Q'(t)Q'(-t\omega) + \omega^{-1}F'\left(-t\omega\right)Q'(t)^2 \\ &- \omega\left(\omega F'\left(t\omega^2\right)Q'\left(-t\omega\right)^2 + F'(t)Q'\left(t\omega^2\right)^2\right) = 0. \end{aligned} \tag{5.1.13}$$

The last term in this equation is precisely $G(t)$ (5.1.11), so we have, indeed, that $G(t)$ is proportional to $Q'(t)$ and we also have the proportionality factor

$$G(t) = \omega^{-2} Q'(t) \left(\omega \Lambda'(t) F'\left(t\omega^2\right) Q'(-t\omega) + Q'(t) F'(-t\omega)\right) \tag{5.1.14}$$

Now we solve Eq. (5.1.10). Let us introduce another useful function

$$\tilde{F}(x) = \prod_{i=1}^{N} (x - z_i) \,. \tag{5.1.15}$$

The former F' function can be expressed through $\tilde{F}$ as

$$F'(x) = (-1)^N \tilde{F}\left(-\sqrt{x}\right) \tilde{F}\left(\sqrt{x}\right) . \tag{5.1.16}$$

The function $G(t)$ becomes

$$G(t) = (-1)^N \left(\omega \tilde{F}\left(\omega\sqrt{t}\right) \tilde{F}\left(-\omega\sqrt{t}\right) Q'\left(-\omega t\right)^2 + \tilde{F}\left(\sqrt{t}\right) \tilde{F}\left(\sqrt{-t}\right) Q'\left(\omega^2 t\right)^2\right) . \tag{5.1.17}$$

Now plug in (5.1.17) the following Ansatz for $Q'(t)$

$$Q'(t) = \mathrm{const}\sqrt{t}\left(\tilde{F}\left(-\sqrt{t}\right)\tilde{F}\left(-\omega\sqrt{t}\right) - \tilde{F}\left(\sqrt{t}\right)\tilde{F}\left(\omega\sqrt{t}\right)\right) . \tag{5.1.18}$$

The overall constant here will be fixed later. After a little bit of algebra we get

$$\begin{aligned} G(t) = {} & \mathrm{const}(-1)^N t\omega^2 \left(\tilde{F}\left(-\sqrt{t}\right)\tilde{F}\left(-\omega\sqrt{t}\right) - \tilde{F}\left(\sqrt{t}\right)\tilde{F}\left(\omega\sqrt{t}\right)\right) \\ & \times \left(\tilde{F}\left(-\sqrt{t}\right)\tilde{F}\left(\omega\sqrt{t}\right) - \tilde{F}\left(\sqrt{t}\right)\tilde{F}\left(-\omega\sqrt{t}\right)\right)\tilde{F}\left(-\omega^2\sqrt{t}\right)\tilde{F}\left(\omega^2\sqrt{t}\right) \end{aligned} \tag{5.1.19}$$

Comparing this to (5.1.18) we see that $G(t)$ is indeed proportional to $Q'(t)$, therefore the BAE are solved with the Q'-function given by the expression (5.1.18). The expression (5.1.18) defines Q' up to an overall constant. Using the definition of $\tilde{F}$ (5.1.15) we can write the Q'-function as a polynomial

$$Q'(t) = \frac{\sqrt{t}\left(\prod_{i=1}^{N}\left(z_i + \sqrt{t}\right)\left(z_i + \sqrt{t}\omega\right) - \prod_{i=1}^{N}\left(\sqrt{t} - z_i\right)\left(\sqrt{t}\omega - z_i\right)\right)}{2\omega^{N+1} E_1(z_1,..,z_N)\left(\omega^{-1} - \omega\right)}, \tag{5.1.20}$$

where we fixed the overall constant. This expression is, in fact, a polynomial in t as will be seen below. Notice that in the polynomial in the parenthesis (which is a polynomial in $\sqrt{t}$) the constant term is absent, therefore its expansion starts with $\sqrt{t}$. This means that $Q'(t)$ starts from the first power in t and thus has one root equal to zero as we established earlier. Expand the function (5.1.20) in t

$$Q'(t) = \sum_{i,j=0}^{N} t^{-(i+j-1)/2+N} \frac{E_i(z_1,..,z_N) E_j(z_1,..,z_N) \left((-1)^{i+j+1}\omega^{-j} + \omega^{-i}\right)}{2\left(1-\omega^2\right) E_1(z_1,..,z_N)}, \tag{5.1.21}$$

and rewrite it in the form

$$Q'(t) = -\frac{1}{4E_1^{(z)}\left(1-\omega^2\right)} \sum_{m=0}^{2N-1} t^{N-m/2} \sum_{i=0}^{m+1} E_i^{(z)} E_{-i+m+1}^{(z)} \times \left(-\omega^{i-m-1} + \omega^{i+2m+2} + \omega^{-i+3m+3} - \omega^{-i}\right), \tag{5.1.22}$$

where we assumed $E_i(z_1,..,z_N) = E_i^{(z)}$. The last factor that depends on ω in the summation is equal to zero for all odd m, so we have

$$Q'(t) = -\frac{1}{2E_1^{(z)}\left(1-\omega^2\right)} \sum_{m=0}^{2N-1} t^{N-m} \sum_{i=0}^{2m+2} E_i^{(z)} E_{2m-i+1}^{(z)} (\omega^{4m+2}\omega^i - \omega^{-i}). \tag{5.1.23}$$

On the other hand, from the definition of Q' in (5.1.7) we have

$$Q'(t) = \prod_{i=1}^{N} (t - \lambda_i) = \sum_{m=0}^{N} t^{N-m} (-1)^m E_m^{(\lambda)}, \tag{5.1.24}$$

where again we assumed $E_i(\lambda_1,..,\lambda_N) = E_i^{(\lambda)}$. The Q-function is the generating function of the Bethe roots exactly in the sense of the formula (5.1.24). Comparing (5.1.24) with (5.1.23) we find the elementary symmetric polynomials in the Bethe roots $E_i^{(\lambda)}$ as functions of the elementary symmetric polynomials in the inhomogeneities $E_i^{(z)}$

$$E_m^{(\lambda)} = -\frac{(-1)^m}{2E_1^{(z)}\left(1-\omega^2\right)} \sum_{i=0}^{2m+2} E_i^{(z)} E_{2m-i+1}^{(z)} (\omega^{4m+2}\omega^i - \omega^{-i}). \tag{5.1.25}$$

To conclude this section we write the eigenvalues of “spin”-k transfer matrices Λ_k. Using Eqs. (5.1.14), (5.1.18) and (5.1.19) and after some algebraic manipulations we find

$$\Lambda(t) = \Lambda_1(t) = \frac{\omega^2 \left(\tilde{F}\left(-t\omega\right) \tilde{F}\left(-t\omega^2\right) + \tilde{F}\left(t\omega\right) \tilde{F}\left(t\omega^2\right)\right)}{\tilde{F}\left(-t\omega\right) \tilde{F}\left(t\omega\right)}. \tag{5.1.26}$$

This is one of the two required initial conditions for the TT-system. The second one will be $\Lambda_0 = 1$. Now we can use the TT relations from (4.5.31), which we write as

$$\Lambda_k(t) = \frac{(-1)^k \tilde{F}\left(-t\omega^{1-k}\right) \tilde{F}\left(t\omega^{1-k}\right) \Lambda_{k-1}(t\omega) + \tilde{F}(-t)\tilde{F}(t)\Lambda_{k-1}(t)\Lambda_{k-1}\left(-t\omega^2\right)}{\tilde{F}(-t)\tilde{F}(t)\Lambda_{k-2}\left(-t\omega^2\right)}. \tag{5.1.27}$$

The higher “spin” eigenvalues are

$$\Lambda_2(t) = \frac{\tilde{F}(-t)\tilde{F}\left(-t\omega^2\right)\tilde{F}(-t\omega)^2 + \tilde{F}(t)\tilde{F}(t\omega)^2\tilde{F}\left(t\omega^2\right)}{\tilde{F}(-t)\tilde{F}(t)\tilde{F}(-t\omega)\tilde{F}(t\omega)},$$

and

$$\Lambda_3(t) = -\frac{\tilde{F}(-t)^2\tilde{F}(-t\omega)^2 - \tilde{F}(-t)\tilde{F}(t)\tilde{F}(t\omega)\tilde{F}(-t\omega) + \tilde{F}(t)^2\tilde{F}(t\omega)^2}{\tilde{F}(-t)\tilde{F}(t)\tilde{F}(-t\omega)\tilde{F}(t\omega)},$$

and so on

$$
\begin{aligned}
\Lambda_4(t) &= \frac{\left(\tilde{F}(-t)\tilde{F}(-t\omega)-\tilde{F}(t)\tilde{F}(t\omega)\right)\left(\tilde{F}(-t)\tilde{F}\left(t\omega^2\right)-\tilde{F}(t)\tilde{F}\left(-t\omega^2\right)\right)}{\tilde{F}(-t)\tilde{F}(t)\tilde{F}(-t\omega)\tilde{F}(t\omega)},\\
\Lambda_5(t) &= \dots
\end{aligned}
$$

The system (5.1.27) can probably be linearized and solved in a closed form.

5.2 The ground state eigenvector

In this section we write the components of the ground state for small systems. The ground state eigenvector of the transfer matrix can be written either using the Tarasov's formula (1.2.6) or using the exponential representation (1.3.5). The raw ground state written in terms of the Bethe roots ζ_i already for $N=2$ is a large expression, therefore we do not present it here. Using the knowledge of the Q-function from Section 5.1 we can eliminate the Bethe roots. After this procedure we find

$$
\begin{aligned}
\psi_{\uparrow,\downarrow}(z_1,z_2) &= -\omega^{3/2} z_1 g(z_1,z_2),\\
\psi_{0,0}(z_1,z_2) &= -\omega\left(z_1+z_2\right) g(z_1,z_2),\\
\psi_{\downarrow,\uparrow}(z_1,z_2) &= -\omega^{1/2} z_2 g(z_1,z_2),
\end{aligned}
$$

where we used the notation for the components from (5.1.5) and the overall factor $g(z_1,z_2)$ depends on the Q-function and reads

$$
g(z_1,z_2) = \frac{\prod_{i=1}^{2}\prod_{j=1}^{2}\left(\zeta_i+\omega z_j\right)\left(\omega\zeta_i-z_j\right)}{3\zeta_1\zeta_2 z_1^3 z_2^3\left(\omega z_2+z_1\right)}.
$$

The dual state has components denoted by $\bar{\psi}$ which have the common factor denoted by $\bar{g}(z_1,z_2)$

$$
\begin{aligned}
\bar{\psi}_{\uparrow,\downarrow}(z_1,z_2) &= \omega^{-1/2}\bar{g}(z_1,z_2),\\
\bar{\psi}_{0,0}(z_1,z_2) &= \bar{g}(z_1,z_2),\\
\bar{\psi}_{\downarrow,\uparrow}(z_1,z_2) &= \omega^{1/2}\bar{g}(z_1,z_2),\\
\bar{g}(z_1,z_2) &= -\frac{\omega\left(\omega z_2+z_1\right)}{3\zeta_1\zeta_2\left(z_1+z_2\right)}.
\end{aligned}
$$

The ground state components for $N=3$ normalized by the entry $\psi_{0,0,0}$ are the following

$$\psi_{\uparrow,0,\downarrow}(z_1,z_2,z_3) = \frac{z_1\left(\omega z_2^2 + \omega z_1 z_2 + 2\omega z_3 z_2 + \omega z_1 z_3 - z_3 z_2\right)}{\sqrt{\omega}\left(z_1+z_2+z_3\right)\left(z_1 z_2 + z_3 z_2 + z_1 z_3\right)},$$
$$\psi_{\uparrow,\downarrow,0}(z_1,z_2,z_3) = \frac{z_1\left(\omega z_3^2 + \omega z_1 z_3 + \omega z_2 z_3 + \omega z_1 z_2 + z_2 z_3\right)}{\sqrt{\omega}\left(z_1+z_2+z_3\right)\left(z_1 z_2 + z_3 z_2 + z_1 z_3\right)},$$
$$\psi_{0\uparrow,\downarrow}(z_1,z_2,z_3) = \frac{z_2\left(\omega z_1^2 + \omega z_2 z_1 + \omega z_3 z_1 + \omega z_2 z_3 + z_3 z_1\right)}{\sqrt{\omega}\left(z_1+z_2+z_3\right)\left(z_1 z_2 + z_3 z_2 + z_1 z_3\right)},$$
$$\psi_{0,\downarrow,\uparrow}(z_1,z_2,z_3) = \frac{z_3\left(\omega z_2 z_1 + z_1^2 + z_2 z_1 + z_3 z_1 + z_2 z_3\right)}{\sqrt{\omega}\left(z_1+z_2+z_3\right)\left(z_1 z_2 + z_3 z_2 + z_1 z_3\right)},$$
$$\psi_{\downarrow,\uparrow,0}(z_1,z_2,z_3) = \frac{z_2\left(\omega z_1 z_3 + z_3^2 + z_1 z_3 + z_2 z_3 + z_1 z_2\right)}{\sqrt{\omega}\left(z_1+z_2+z_3\right)\left(z_1 z_2 + z_3 z_2 + z_1 z_3\right)},$$
$$\psi_{\downarrow,0,\uparrow}(z_1,z_2,z_3) = \frac{z_3\left(-\omega z_1 z_2 + z_2^2 + 2 z_1 z_2 + z_3 z_2 + z_1 z_3\right)}{\sqrt{\omega}\left(z_1+z_2+z_3\right)\left(z_1 z_2 + z_3 z_2 + z_1 z_3\right)}.$$

The dual state normalised in a similar way is again trivial

$$\bar{\psi}_{\uparrow,0,\downarrow}(z_1,z_2,z_3) = \bar{\psi}_{\uparrow,\downarrow,0}(z_1,z_2,z_3) = \bar{\psi}_{0\uparrow,\downarrow}(z_1,z_2,z_3) = \omega^{-1/2},$$
$$\bar{\psi}_{0,\downarrow,\uparrow}(z_1,z_2,z_3) = \bar{\psi}_{\downarrow,\uparrow,0}(z_1,z_2,z_3) = \bar{\psi}_{\downarrow,0,\uparrow}(z_1,z_2,z_3) = \omega^{1/2}.$$

We also present here the small size scalar product result. Recall the definition of the scalar product (1.4.5). S_N depends on the three sets of variables: the Bethe roots $\zeta_1,..,\zeta_N$, the parameters $\mu_1,..,\mu_N$ and the inhomogeneities. The expression for S_2 is

$$\begin{aligned}
&S_2(\mu_1,\mu_2,\zeta_1(z_1,z_2),\zeta_2(z_1,z_2);z_1,z_2) = h_2(\mu_1,\mu_2,\zeta_1,\zeta_2;z_1,z_2)\\
&\times E_2^{(z)}\left(E_2^{(z)} - {E_1^{(z)}}^2\right)\Big(-{E_1^{(\mu^2)}}^2 E_2^{(z)} + E_1^{(\mu^2)}{E_1^{(z)}}^2 E_2^{(z)} - 2E_2^{(\mu^2)}E_2^{(z)}\\
&+{E_1^{(\mu^2)}}^2{E_1^{(z)}}^2 + E_2^{(\mu^2)}{E_1^{(z)}}^2 + 2E_1^{(\mu^2)}E_2^{(\mu^2)} + {E_2^{(\mu^2)}}^3\Big),
\end{aligned}$$

where $E_i^{(\mu^2)} = E_i(\mu_1^2,\mu_2^2)$ and the overall factor is

$$\begin{aligned}
h_2(\mu_1,\mu_2,\zeta_1,\zeta_2;z_1,z_2) = &-\frac{1}{243\zeta_1^2\zeta_2^2\mu_1^2\mu_2^2 z_1^7 z_2^7}\prod_{i,j=1}^{2}\frac{\omega^3\left(\omega\zeta_i^2+\zeta_j^2\right)\left(\omega\mu_i^2+\mu_j^2\right)}{\left(\omega^2-1\right)^2\zeta_i\mu_i\zeta_j\mu_j}\\
\times\prod_{1\le i<j\le 2}&\frac{z_j^6}{\left(\omega^2-1\right)^2\zeta_i\mu_i\left(\omega^3\zeta_i^2+z_j^2\right)\left(\omega^3\mu_i^2+z_j^2\right)}.
\end{aligned}$$

5.3 Conclusion

We saw that in the case when $q^3 = -1$ it is possible to obtain the formulae for the symmetric polynomials of the Bethe roots. The ground state components can then be derived from the algebraic Bethe Ansatz. The next question would be how to compute the correlation functions. One way of doing it is by using the scalar products [84]. If one finds a good formula for the scalar products, then we could use the knowledge of the Q-function to rid of the Bethe roots. This would lead to the form factors. An example of such computation appears in [56] for the XXZ spin chain.

The calculations of this chapter were performed for the ground state at $q^3 = -1$. It is possible to repeat the same for the case $q^4 = -1$. The resulting expressions are slightly more complicated. Alternatively, the (also conjectured) ground state components at $q^4 = -1$ can be obtained via the vertex operators. A comprehensive and rigorous discussion of the vertex operators approach should be given separately.

Appendix A

$R^{(k)}$-matrix

Coefficients of the $R^{(k)}$-matrix (4.5.28), (4.5.29) are presented below.

$$\begin{aligned}
\lambda_1^{(1)} &= \frac{q^k\left(\zeta\kappa_1^2+q^3\right)\left(q^2-\zeta\kappa_2^2\right)}{\kappa_1\kappa_2},\\
\lambda_2^{(1)} &= (q^2-1)^2\Big(\frac{\left(\zeta+q^3\right)\left(\zeta q^{2k}-1\right)\left(q^2-\zeta\kappa_1^2\right)\left(\zeta\kappa_2^2+q\right)}{\left(q^2-1\right)^2\left(\zeta\kappa_2^2-1\right)\left(\zeta\kappa_1^2+q\right)}\Big)\\
&- \frac{\zeta\kappa_2 q^{k-\frac{1}{2}}D_2\left(\kappa_1,\kappa_2\right)\left(\zeta\kappa_1^2+q^3\right)}{\kappa_1(q+1)\left(\zeta\kappa_2^2-1\right)} - \frac{\zeta\kappa_1 q^{k+\frac{1}{2}}D_1\left(\kappa_1,\kappa_2\right)\left(q^2-\zeta\kappa_2^2\right)}{\kappa_2(q+1)\left(\zeta\kappa_1^2+q\right)},\\
\lambda_2^{(2)} &= (q^2-1)^2\frac{\zeta\kappa_2 q^k}{\kappa_1},\\
\lambda_2^{(3)} &= -(q^2-1)^2\frac{\zeta\kappa_1 q^{k-1}}{\kappa_2},
\end{aligned}$$

$$\begin{aligned}
\lambda_3^{(1)} &= \zeta\left(q^2-1\right)^2\Big(\frac{\kappa_1\kappa_2 q^{-k}\left(q^2-\zeta\right)\left(\zeta+q^3\right)\left(\zeta q^{2k}-1\right)\left(\zeta q^{2k+1}+1\right)}{\zeta\left(q^2-1\right)^2\left(\zeta\kappa_1^2-1\right)\left(\zeta\kappa_2^2 q+1\right)}\\
&- \frac{\left(\zeta q^{2k}-1\right)}{(q+1)}\Big(\frac{\kappa_1^2\left(\zeta+q^3\right)D_1\left(\kappa_1,\kappa_2\right)\left(\zeta\kappa_2^2+q\right)}{\sqrt{q}\left(\zeta\kappa_1^2-1\right)\left(\zeta\kappa_2^2-1\right)\left(\zeta\kappa_1^2 q+1\right)}\\
&+ \frac{\kappa_2^2\sqrt{q}\left(\zeta+q^3\right)D_2\left(\kappa_1,\kappa_2\right)\left(q^2-\zeta\kappa_1^2\right)}{\left(\zeta\kappa_1^2+q\right)\left(\zeta\kappa_2^2 q+1\right)\left(\zeta\kappa_2^2 q^2-1\right)}\Big)\\
&+ \frac{\kappa_1(q-1)^2 q^{k-\frac{1}{2}}D_1\left(\kappa_1,\kappa_2\right)\left(q^2-\zeta\kappa_2^2\right)D_1\left(\kappa_1 q,\kappa_2\right)}{\kappa_2(q+1)\left(\zeta\kappa_1^2 q+1\right)}\\
&+ \frac{\kappa_2(q-1)^2 q^{k-\frac{1}{2}}D_2\left(\kappa_1,\kappa_2\right)\left(\zeta\kappa_1^2+q^3\right)D_2\left(\kappa_1,\kappa_2 q\right)}{\kappa_1(q+1)\left(\zeta\kappa_2^2 q^2-1\right)}\\
&- \frac{\left(\zeta\kappa_2^2-1\right)q^{k-\frac{7}{2}}B_1\left(\kappa_1,\kappa_2\right)D_2\left(\kappa_1,\kappa_2\right)C_1\left(\kappa_1,\kappa_2 q\right)D_1\left(\kappa_1,\kappa_2 q\right)}{\kappa_1\kappa_2(q+1)\left(\zeta\kappa_1\kappa_2-1\right)^2\left(\zeta\kappa_1\kappa_2+1\right)^2\left(\zeta\kappa_1^2+q\right)}
\end{aligned}$$

$$
\begin{aligned}
&- \frac{q^{k-\frac{11}{2}} B_2\left(\kappa_1,\kappa_2\right) D_1\left(\kappa_1,\kappa_2\right)\left(\zeta\kappa_1^2+q\right) C_2\left(\kappa_1,\kappa_2 q\right) D_2\left(\kappa_1 q,\kappa_2\right)}{\kappa_1\kappa_2(q+1)\left(\zeta\kappa_1\kappa_2-1\right)^2\left(\zeta\kappa_1\kappa_2+1\right)^2\left(\zeta\kappa_2^2-1\right)} \\
&+ \frac{\left(\kappa_1^2+\kappa_2^2\right)(q-1)\left(\zeta\kappa_2^2-1\right)\left(\zeta^2\kappa_1^2\kappa_2^2+1\right) q^k\left(q^2-\zeta\kappa_1^2\right)}{\kappa_1\kappa_2(q+1)^2\left(\zeta\kappa_1\kappa_2-1\right)^2\left(\zeta\kappa_1\kappa_2+1\right)^2} \\
&\quad + \frac{(q-1)\left(\zeta^2\kappa_1^2\kappa_2^2+1\right) q^{k-1}\left(\kappa_1^2+\kappa_2^2 q^2\right)\left(\zeta\kappa_1^2+q\right)\left(\zeta\kappa_2^2+q\right)}{\kappa_1\kappa_2(q+1)^2\left(\zeta^2\kappa_1^2\kappa_2^2-1\right)^2} \\
&\quad + \frac{2\zeta(q-1)^2 q^{k-1}\left(q^2-\zeta\kappa_1^2\right)\left(\zeta\kappa_2^2+q\right)}{\kappa_1^{-1}\kappa_2^{-1}(q+1)^2\left(\zeta^2\kappa_1^2\kappa_2^2-1\right)^2} \\
&\quad + \frac{2\zeta\left(\kappa_1^2+\kappa_2^2\right)\left(\zeta\kappa_2^2-1\right) q^k\left(\kappa_1^2+\kappa_2^2 q^2\right)\left(\zeta\kappa_1^2+q\right)}{\kappa_1\kappa_2(q+1)^2\left(\zeta^2\kappa_1^2\kappa_2^2-1\right)^2}\Bigg),
\end{aligned}
$$

$$
\begin{aligned}
\lambda_3^{(2)} &= \zeta\left(q^2-1\right)^2 \frac{G\left(i\kappa_2\sqrt{q},\frac{i\kappa_1}{\sqrt{q}}\right)}{\left(\kappa_2^2+\kappa_1^2 q^5\right)\left(\kappa_1^2+\kappa_2^2 q^3\right)}, \\
\lambda_3^{(3)} &= \zeta\left(q^2-1\right)^2 \frac{G\left(\kappa_1,\kappa_2\right)}{\left(\kappa_2^2+\kappa_1^2 q\right)\left(\kappa_1^2+\kappa_2^2 q^7\right)}, \\
\lambda_3^{(4)} &= \zeta\left(q^2-1\right)^2 \frac{\kappa_2(q-1)^2 q^{k-2}}{\kappa_1}, \\
\lambda_3^{(5)} &= \zeta\left(q^2-1\right)^2 \frac{\kappa_1(q-1)^2 q^{k-3}}{\kappa_2},
\end{aligned}
$$

$$
\begin{aligned}
\nu_1^{(1)} &= q^2\left(q^2-1\right)\left(q^2-\zeta\kappa_2^2\right), \\
\nu_1^{(2)} &= q\left(q^2-1\right)\left(\zeta\kappa_1^2+q^3\right),
\end{aligned}
$$

$$
\begin{aligned}
\nu_2^{(1)} &= (q-1)^2(q+1)q^{1-k}\kappa_1\kappa_2 q\left(q^2-\zeta\kappa_2^2\right), \\
\nu_2^{(2)} &= (q-1)^2(q+1)q^{1-k}\kappa_1\kappa_2\left(\zeta\kappa_1^2+q^3\right), \\
\nu_2^{(3)} &= -(q-1)^2(q+1)q^{1-k} \\
&\times \frac{\kappa_1\kappa_2 q(q+1)\left(\kappa_1^2+\kappa_2^2 q\right)\left(\zeta\kappa_1^2\kappa_2^2\left(q^2+1\right)+q^2\left(\kappa_2^2 q-\kappa_1^2\right)\right)}{\left(\kappa_2^2+\kappa_1^2 q\right)\left(\kappa_1^2-\kappa_2^2 q^2\right)},
\end{aligned}
$$

$$
\begin{aligned}
\nu_3^{(1)} &= -(q^2-1)\Bigg(\frac{\zeta\kappa_2^2(q-1)B_1\left(\kappa_1,\kappa_2\right) D_2\left(\kappa_1,\kappa_2\right)}{\sqrt{q}\left(\zeta^2\kappa_1^2\kappa_2^2-1\right)} \\
&+ \frac{\zeta\kappa_2^2(q-1)B_2\left(\kappa_1,\kappa_2\right)\left(\zeta\kappa_1^2+q\right) D_2\left(\kappa_1 q,\kappa_2\right)}{\sqrt{q}\left(\zeta\kappa_2^2-1\right)\left(\zeta^2\kappa_1^2\kappa_2^2-1\right)} \\
&- \frac{\zeta\kappa_1^2(q-1)^2\sqrt{q}\left(q^2-\zeta\kappa_2^2\right) D_1\left(\kappa_1 q,\kappa_2\right)}{\zeta\kappa_1^2 q+1} \\
&+ \frac{\kappa_1\kappa_2 q^{-k}\left(\zeta+q^3\right)\left(\zeta q^{2k}-1\right)\left(\zeta\kappa_2^2+q\right)}{\left(\zeta\kappa_2^2-1\right)\left(\zeta\kappa_1^2 q+1\right)}\Bigg),
\end{aligned}
$$

$$
\begin{aligned}
\nu_3^{(2)} &= (q^2-1)\Bigg(\frac{\zeta\kappa_1^2(q-1)B_2\left(\kappa_1,\kappa_2\right)D_1\left(\kappa_1,\kappa_2\right)}{\sqrt{q}\left(\zeta^2\kappa_1^2\kappa_2^2-1\right)} \\
&- \frac{\zeta\kappa_1^2(q-1)\left(\zeta\kappa_2^2-1\right)B_1\left(\kappa_1,\kappa_2\right)D_1\left(\kappa_1,\kappa_2 q\right)}{\sqrt{q}\left(\zeta^2\kappa_1^2\kappa_2^2-1\right)\left(\zeta\kappa_1^2+q\right)} \\
&+ \frac{\zeta\kappa_2^2(q-1)^2\sqrt{q}\left(\zeta\kappa_1^2+q^3\right)D_2\left(\kappa_1,\kappa_2 q\right)}{1-\zeta\kappa_2^2q^2} \\
&+ \frac{\kappa_1\kappa_2 q^{-k}\left(\zeta+q^3\right)\left(\zeta q^{2k}-1\right)\left(q^2-\zeta\kappa_1^2\right)}{\left(\zeta\kappa_1^2+q\right)\left(\zeta\kappa_2^2q^2-1\right)}\Bigg), \\
\nu_3^{(3)} &= (q^2-1)\frac{\zeta\kappa_2^2(q-1)^2(q+1)}{q}, \\
\nu_3^{(4)} &= -(q^2-1)\frac{\zeta\kappa_1^2(q-1)^2(q+1)}{q^2},
\end{aligned}
$$

$$
\begin{aligned}
\mu_1^{(1)} &= \frac{q^{1-2k}}{\zeta\left(q^2-1\right)}\frac{\zeta\kappa_2^2-q^2}{\kappa_2^2}, \\
\mu_1^{(2)} &= \frac{q^{1-2k}}{\zeta\left(q^2-1\right)}\frac{\zeta\kappa_1^2+q^3}{\kappa_1^2}),
\end{aligned}
$$

$$
\begin{aligned}
\mu_2^{(1)} &= \zeta(q-1)^2(q+1)q^{3k}\frac{q^2-\zeta\kappa_2^2}{\kappa_1\kappa_2^3}, \\
\mu_2^{(2)} &= -\zeta(q-1)^2(q+1)q^{3k}\frac{\zeta\kappa_1^2+q^3}{\kappa_1^3\kappa_2}, \\
\mu_2^{(3)} &= -\zeta(q-1)^2(q+1)q^{3k} \\
&\times \frac{(q+1)\left(\kappa_1^2+\kappa_2^2q\right)\left(\zeta\kappa_1^2+q^5+q^3-\zeta\kappa_2^2q\right)}{\kappa_1\kappa_2\left(\kappa_1-\kappa_2q\right)\left(\kappa_1+\kappa_2q\right)\left(\kappa_2^2+\kappa_1^2q\right)},
\end{aligned}
$$

$$
\begin{aligned}
\mu_3^{(1)} &= \zeta(q-1)^2(q+1)q^{3k}\Bigg(\frac{q^{\frac{3}{2}-k}C_1\left(\frac{\kappa_1}{q},\kappa_2q\right)D_2\left(\frac{\kappa_1}{q},\kappa_2\right)}{\kappa_1^2\kappa_2^2\left(q^2-\zeta^2\kappa_1^2\kappa_2^2\right)} \\
&- \frac{q^{-k-\frac{5}{2}}D_2\left(\kappa_1,\kappa_2\right)\left(\zeta\kappa_1^2+q^3\right)C_2\left(\frac{\kappa_1}{q},\kappa_2q\right)}{\kappa_1^2\kappa_2^2\left(\zeta\kappa_2^2-1\right)\left(q^2-\zeta^2\kappa_1^2\kappa_2^2\right)} + \frac{(q-1)q^{\frac{1}{2}-k}D_1\left(\kappa_1,\kappa_2\right)\left(q^2-\zeta\kappa_2^2\right)}{\kappa_2^2\left(\zeta\kappa_1^2+q\right)} \\
&+ \frac{\kappa_1q^{-2k}\left(\zeta+q^3\right)\left(\zeta q^{2k}-1\right)\left(\zeta\kappa_2^2+q\right)}{\kappa_2(q-1)\left(\zeta\kappa_2^2-1\right)\left(\zeta\kappa_1^2+q\right)}\Bigg),
\end{aligned}
$$

$$
\begin{aligned}
\mu_3^{(2)} &= \zeta(q-1)^2(q+1)q^{3k}\Bigg(\frac{q^{\frac{1}{2}-k}C_1\left(\kappa_1,\kappa_2\right)D_1\left(\kappa_1,\kappa_2\right)\left(q^2-\zeta\kappa_2^2\right)}{\kappa_1^2\kappa_2^2\left(\zeta\kappa_1^2+q\right)\left(q^2-\zeta^2\kappa_1^2\kappa_2^2\right)} \\
&+ \frac{q^{\frac{1}{2}-k}C_2\left(\kappa_1,\kappa_2\right)D_1\left(\kappa_1,\frac{\kappa_2}{q}\right)}{\kappa_1^2\kappa_2^2\left(q^2-\zeta^2\kappa_1^2\kappa_2^2\right)} + \frac{(q-1)q^{-k-\frac{1}{2}}D_2\left(\kappa_1,\kappa_2\right)\left(\zeta\kappa_1^2+q^3\right)}{\kappa_1^2\left(\zeta\kappa_2^2-1\right)}
\end{aligned}
$$

$$+\frac{\kappa_2 q^{-2k}\left(\zeta+q^3\right)\left(\zeta q^{2k}-1\right)\left(\zeta\kappa_1^2-q^2\right)}{\kappa_1(q-1)\left(\zeta\kappa_2^2-1\right)\left(\zeta\kappa_1^2+q\right)}\Bigg),$$

$$\mu_3^{(3)}=\zeta(q-1)^2(q+1)q^{3k}\frac{\left(q^2-1\right)q^{-k-1}}{\kappa_2^2},$$
$$\mu_3^{(4)}=-\zeta(q-1)^2(q+1)q^{3k}\frac{\left(q^2-1\right)q^{-k}}{\kappa_1^2}+\bar{a}_2 a_1^2.$$

These expressions are written in terms of rescaled operators

$$\bar{a}_i=\sqrt{q^{1/2}+q^{-1/2}}\tilde{\bar{a}}_i,\quad a_i=\sqrt{q^{1/2}+q^{-1/2}}\tilde{a}_i,$$

and the tilde was omitted. Also we used several functions $B_1(\kappa_1,\kappa_2)$, $B_2(\kappa_1,\kappa_2)$ and $C_1(\kappa_1,\kappa_2)$, $C_2(\kappa_1,\kappa_2)$ which are defined as

$$B_1\left(\kappa_1,\kappa_2\right)=\zeta^2\kappa_1^4+\zeta\kappa_1^2+\zeta\kappa_2^2q^3+q^3+\zeta^2\kappa_2^2\kappa_1^2q^2-q^2,$$
$$B_2\left(\kappa_1,\kappa_2\right)=-\zeta\kappa_2^2+q^2+\zeta^2\kappa_2^4q+\zeta^2\kappa_1^2\kappa_2^2q-\zeta\kappa_1^2q-q,$$
$$C_1\left(\kappa_1,\kappa_2\right)=\zeta^2\kappa_2^2\kappa_1^4+\zeta\kappa_2^2\kappa_1^2q^3+\kappa_1^2q^3+\kappa_2^2q^3+\zeta\kappa_1^4q^2-q\zeta^2\kappa_2^2\kappa_1^4,$$
$$C_2\left(\kappa_1,\kappa_2\right)=-\zeta^2\kappa_1^2\kappa_2^4-\kappa_1^2q^5+\zeta\kappa_1^2\kappa_2^2q^4-\kappa_2^2q^3+\zeta^2\kappa_1^2\kappa_2^4q+\zeta\kappa_2^4q.$$

The functions $D_1(\kappa_1\kappa_2)$ and $D_2(\kappa_1\kappa_2)$ were defined earlier (4.5.7)-(4.5.8).

Appendix B

Factorisation of the R-matrix

The weights of the $R(z_1/z_2)$-matrix are such that when the ratio $z_1/z_2 = q^{-2}$ we can write it in a product form

$$\check{R}(q^{-2}) = Y^T Y. \tag{B.1}$$

This is related to the quasi triangularity condition of the R-matrix (4.1.2). In this appendix we give an elementary presentation of the factorisation property (B.1).

Y^T maps $\mathbb{C}^3$ to $\mathbb{C}^3 \otimes \mathbb{C}^3$ and Y maps $\mathbb{C}^3 \otimes \mathbb{C}^3$ to $\mathbb{C}^3$. Using v_a as a standard basis of $\mathbb{C}^3$ we can write the components of Y^T and Y

$$Y^T = \sum_{a,b,c} \tilde{y}^c_{a,b} v_a \otimes e_{b,c}, \qquad Y = \sum_{a,b,c} y^{b,c}_a e_{a,b} \otimes v_c. \tag{B.2}$$

Components of the Y-matrices are depicted graphically in Fig. B.1 and Fig. B.2.

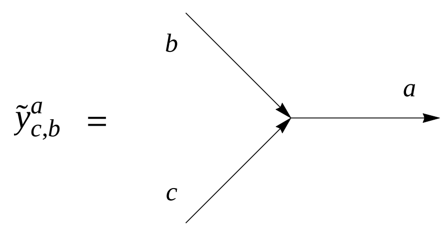


FIGURE B.1 – Graphical representation of the components $\tilde{y}^a_{c,b}$.

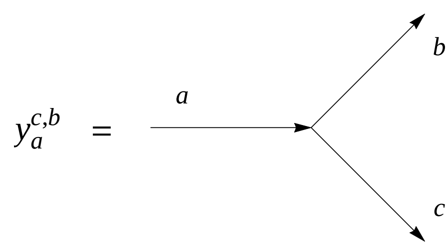


FIGURE B.2 – Graphical representation of the components $y^{c,b}_a$.

The relation between components $\check{r}(q^{-2})$, $\tilde{y}$ and y is

$$\check{r}^{c_1,b_1}_{c,b}(q^{-2}) = \sum_a \tilde{y}^a_{c,b} y^{c_1,b_1}_a. \tag{B.3}$$

This means that we take the vertex of the $\check{R}$-matrix turn it into a line and then cut it in the middle, see Fig. B.3. This produces the trivalent vertices which are the two

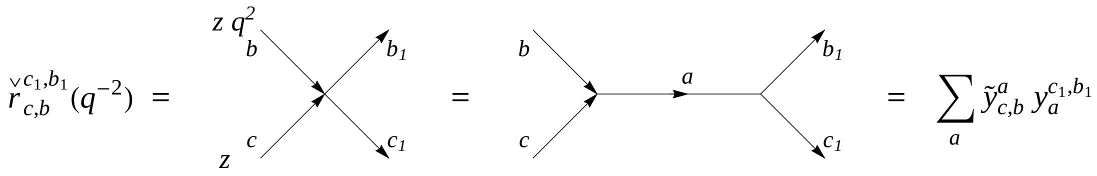


FIGURE B.3 – The splitting of the $\check{R}$ matrix into Y^T and Y.

Y-matrices. The nontrivial entries of Y^T are those in which the indices are related as in $\tilde{y}^{a+b}_{a,b} = \tilde{y}_{a,b}$, so we have

$$\begin{pmatrix} 0 & 0 & 0 \\ \tilde{y}_{1,0} & 0 & 0 \\ 0 & \tilde{y}_{1,-1} & 0 \\ \tilde{y}_{0,1} & 0 & 0 \\ 0 & \tilde{y}_{0,0} & 0 \\ 0 & 0 & \tilde{y}_{0,-1} \\ 0 & \tilde{y}_{-1,1} & 0 \\ 0 & 0 & \tilde{y}_{-1,0} \\ 0 & 0 & 0 \end{pmatrix}. \tag{B.4}$$

And for Y, $y^{a,b}_{a+b} = y_{a,b}$

$$\begin{pmatrix} 0 & y_{1,0} & 0 & y_{0,1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & y_{1,-1} & 0 & y_{0,0} & 0 & y_{-1,1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & y_{0,-1} & 0 & y_{-1,0} & 0 \end{pmatrix}, \tag{B.5}$$

Many of these entries follow from Eq. (B.1). Other entries will be fixed by the Yang–Baxter relation, but first, we write the “unitarity” condition for Y’s

$$YY^T = \tilde{I},$$

$$\sum_{b,c} y^{b,c}_{a_1} \tilde{y}^{a_2}_{c,b} = \tilde{I}_{a_1,a_2} = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -\frac{q^4+q^3-2q^2+q+1}{q^2} & 0 \\ 0 & 0 & -2 \end{pmatrix}. \tag{B.6}$$

Where we used the identity

$$\sum_a f_a v_{b_1} \otimes e_{c_1,a} \cdot e_{a,c_2} \otimes v_{a_2} = e_{b_1,c_2} \otimes e_{c_1,a_2} \sum_a f_a. \tag{B.7}$$

Graphically the unitarity for Y operators is shown in Fig. B.4. The Yang–Baxter

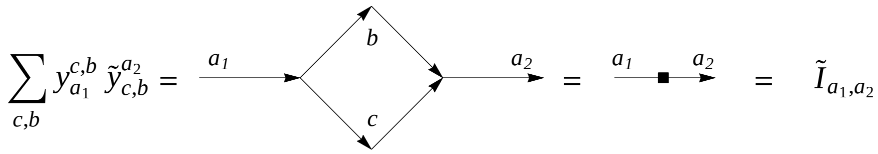


FIGURE B.4 – Unitarity relation for Y^T and Y.

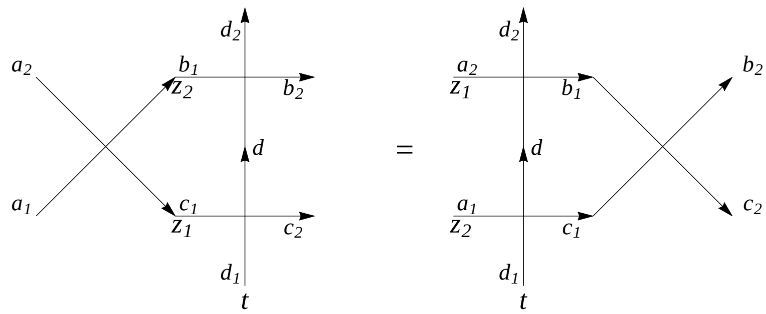


FIGURE B.5 – The Yang–Baxter equation with the $\check{R}$.

equation with the $\check{R}$-matrix is shown in Fig. B.5. If we set $z_1 = z_2 q^{-2}$ in Fig. B.5 and attach the Y matrix from the left sides in Fig. (B.5) we get Fig. B.6 (where we omit the indices, spectral parameters and the orientations of the vector spaces).

From the equation in Fig. B.6, if Y and Y^T satisfy

$$YRRY^T = \text{const}\tilde{I}R, \tag{B.8}$$

which is better seen in Fig. B.7, we obtain the equation

$$YRR = \text{const}RY, \tag{B.9}$$

which is shown graphically in Fig. B.8. In components Eq. (B.8) reads

$$\sum_{c_1,b_1,c_2,b_2,d} y_{a_1}^{c_1,b_1} r_{c_1,d_1}^{c_2,d}(zq^{-2}/t)\, r_{b_1,d}^{b_2,d_2}(z/t)\, \tilde{y}_{b_2,c_2}^{a_2} = \text{const} \sum_i \tilde{I}_{a_1,i} r_{i,d_1}^{a_2,d_2}(-q^{-1}z/t), \tag{B.10}$$

where "const" is the proportionality factor in Fig. B.7 and Fig. B.8

$$\text{const} = \frac{q(t-z)(t+q^5 z)}{(q+1)t^2}. \tag{B.11}$$

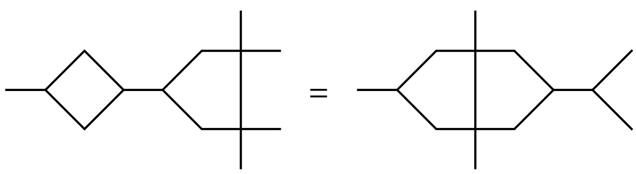


FIGURE B.6 – Attaching Y to the Yang–Baxter equation Fig. B.5 where $z_1 = z_2 q^{-2}$.

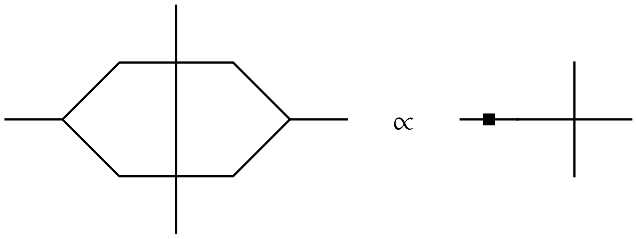


Figure B.7 – Pushing the line through a trivalent vertex and using the unitarity for Y and Y^T.

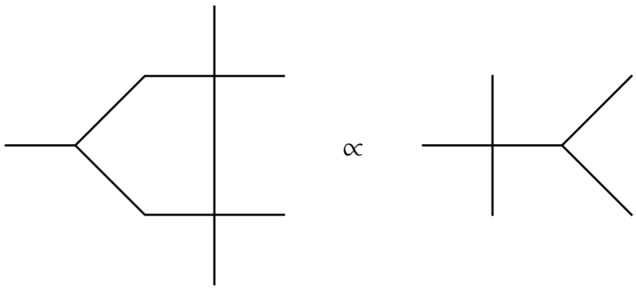


Figure B.8 – Eq. (B.9).

Eq. (B.1) and Eq. (B.8) fix the entries of Y^T and Y up to a normalization

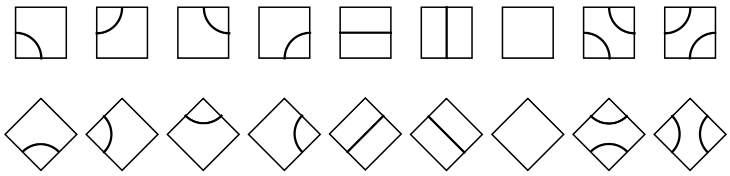
$$Y^T = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -\frac{2iq^2}{(q-4)q+1} & 0 \\ -1 & 0 & 0 \\ 0 & \frac{2i(q-1)\sqrt{q}}{(q-4)q+1} & 0 \\ 0 & 0 & -1 \\ 0 & \frac{2i}{(q-4)q+1} & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}. \tag{B.12}$$

$$Y = \begin{pmatrix} 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{i\left(q^2-4q+1\right)}{2q^2} & 0 & \frac{i(q-1)\left(q^2-4q+1\right)}{2q^{3/2}} & 0 & -\frac{1}{2}i\left(q^2-4q+1\right) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \end{pmatrix}. \tag{B.13}$$

Appendix C

The loop basis

In the loop basis we have the face operators



$\rho^{(1)}$ $\rho^{(2)}$ $\rho^{(3)}$ $\rho^{(4)}$ $\rho^{(5)}$ $\rho^{(6)}$ $\rho^{(7)}$ $\rho^{(8)}$ $\rho^{(9)}$

FIGURE C.1 – The plaquettes here are the operators acting in the so-called space of the link pattern states [54, 57]. The R°-matrix is a linear combination of the operators from the firs row. The $\check{R}^{\circ}$-matrix is a linear combination of the operators appearing in the second row. The latter operators are denoted by $\rho^{(i)}$, respectively, as shown in the last row.

The corresponding R-matrices in the loop basis will be distinguished by the superscript $\circ$. The $\check{R}^{\circ}$-matrix is

$$\check{R}^{\circ}_j(z_j,z_{j+1}) = \sum_{i=1}^{9} \rho_j^{(i)} r_i(z_j,z_{j+1}). \tag{C.1}$$

There is a mapping of the weights of the nineteen vertex model to the weights $r_i(z)$ [64]. First step is to consider the tilted R-matrices Fig. C.2 for both vertex and loop versions. Now, take the tilted loop R-matrix and orient each loop in the plaquettes. The orientation has a weight of $e^{i\nu\phi}$, where ν is a free parameter for now and ϕ is a clockwise turning angle of the loop, considering that each loop enters and exits the rhombic plaquette perpendicularly. Next, we need to fix the appropriate grading for the vertex model R-matrix to make this map possible. With our conventions this is achieved by performing the following transform of the vertex model R-matrix

$$\bar{R}(z_1,z_2) = G(z_1) \otimes G(z_2) R(z_1,z_2) G^{-1}(z_1) \otimes G^{-1}(z_2), \tag{C.2}$$

where the matrix $G(z)$ is the following diagonal matrix

$$G(z) = \begin{pmatrix} z^l & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & z^{-l} \end{pmatrix}, \tag{C.3}$$

which coincides with $G(z^l)$ from the former definition (4.4.31).

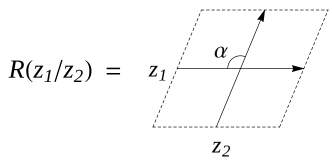


FIGURE C.2 – The tilted R-matrix. The dashed rhombus is the dual representation of the R-matrix corresponding to the R°-matrix.

This transform affects the weights x_5, x_6, x_7, y_5, y_6 and y_7 in following way

$$\begin{aligned} &\bar{x}_5(\zeta) = \zeta^l x_5(\zeta), \quad \bar{x}_6(\zeta) = \zeta^l\, x_6(\zeta), \quad \bar{x}_7(\zeta) = \zeta^{2l} x_7(\zeta), \\ &\bar{y}_5(\zeta) = \zeta^{-l} y_5(\zeta), \quad \bar{y}_6(\zeta) = \zeta^{-l} y_6(\zeta), \quad \bar{y}_7(\zeta) = \zeta^{-2l} y_7(\zeta), \end{aligned} \tag{C.4}$$

For the rest of the gauge transformed weights we have: $\bar{x}_i = x_i$ and $\bar{y}_i = y_i$. The change of the grading introduced a new parameter l which will be very important in what follows. The correspondence of the loop and the vertex weights is as follows

$$\begin{aligned} &\bar{x}_1(\zeta) = r_9(\zeta), \quad \bar{x}_2(\zeta) = r_5(\zeta) = r_6(\zeta), \\ &\bar{x}_3(\zeta) = r_8(\zeta), \quad \bar{x}_4(\zeta) = r_7(\zeta), \\ &\bar{x}_5(\zeta) = r_2(\zeta) e^{-i\nu\alpha} = r_4(\zeta) e^{-i\nu\alpha}, \\ &\bar{x}_6(\zeta) = r_1(\zeta) e^{-i\nu(\alpha-\pi)} = r_3(\zeta) e^{-i\nu(\alpha-\pi)}, \\ &\bar{y}_5(\zeta) = r_2(\zeta) e^{i\nu\alpha} = r_4(\zeta) e^{i\nu\alpha}, \\ &\bar{y}_6(\zeta) = r_1(\zeta) e^{i\nu(\alpha-\pi)} = r_3(\zeta) e^{i\nu(\alpha-\pi)}, \\ &\bar{x}_7(\zeta) = r_8(\zeta) e^{-2i\nu(\alpha-\pi)} + r_9(\zeta) e^{-2i\nu\alpha}, \\ &\bar{y}_7(\zeta) = r_8(\zeta) e^{2i\nu(\alpha-\pi)} + r_9(\zeta) e^{2i\nu\alpha}. \end{aligned} \tag{C.5}$$

Integrability (YB equation) requires that $e^{i\nu\alpha} = \zeta^{-l}$ and $e^{i\nu\pi} = -iq^{-1}$. In the loop model a closed loop has a weight denoted by n. Using the above notion of oriented loops we have $n = e^{2i\nu\pi} + e^{-2i\nu\pi}$, which in terms of q reads $n = -q^2 - q^{-2}$. Eqs. (C.5) define the mapping between the weights of the loop model and the vertex model. Previously when we worked with the loop models [54, 57, 58] we used different conventions, in particular, we were using $\tilde{q} = -q$ for the parameter q and also we used the weights

$\tilde{r}_i(z_1,z_2) = z_2^2 r_i(z_1^2/z_2^2)$. The loop model weights read

$$\begin{aligned}
\tilde{r}_1(z_1,z_2) &= \tilde{r}_3(z_1,z_2) = \tilde{q}^{3/2}\left(1-\tilde{q}^2\right)\left(z_2^2-z_1^2\right),\\
\tilde{r}_2(z_1,z_2) &= \tilde{r}_4(z_1,z_2) = \left(\tilde{q}^2-1\right)\left(\tilde{q}^3 z_2^2 - z_1^2\right),\\
\tilde{r}_5(z_1,z_2) &= r_6(z_1,z_2) = \frac{\tilde{q}\,(z_1^2-z_2^2)\,(\tilde{q}^3 z_1^2 - z_2^2)}{z_1 z_2},\\
\tilde{r}_7(z_1,z_2) &= z_1 z_2\left(-\frac{\tilde{q}^4 z_2^2}{z_1^2} + (\tilde{q}+1)\left(\tilde{q}^4-\tilde{q}^2+1\right) - \frac{\tilde{q} z_1^2}{z_2^2}\right),\\
\tilde{r}_8(z_1,z_2) &= -\frac{\tilde{q}^2\,(z_1^2-z_2^2)\,(\tilde{q} z_1^2 - z_2^2)}{z_1 z_2},\\
\tilde{r}_9(z_1,z_2) &= -\frac{(\tilde{q}^2 z_1^2 - z_2^2)\,(\tilde{q}^3 z_1^2 - z_2^2)}{z_1 z_2}.
\end{aligned} \tag{C.6}$$

Now, we need to interpret the Y operators in terms of the loop plaquettes and find the weights of the Y-matrix in the loop basis. Before we do that let us change q to $\tilde{q}$ in the R-matrix, and then apply the gauge transformation to the R-matrix. The Y and Y^T matrices after the transformation become

$$\bar{Y} = \begin{pmatrix} 0 & \tilde{q}^l & 0 & \tilde{q}^{-l} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & i\tilde{q}^{2l+1} & 0 & \frac{\tilde{q}+1}{\sqrt{\tilde{q}}} & 0 & -i\tilde{q}^{-2l-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \tilde{q}^l & 0 & \tilde{q}^{-l} & 0 \end{pmatrix}, \tag{C.7}$$

$$\bar{Y}^T = \begin{pmatrix} 0 & 0 & 0 \\ \tilde{q}^l & 0 & 0 \\ 0 & i\tilde{q}^{2l+1} & 0 \\ \tilde{q}^{-l} & 0 & 0 \\ 0 & \frac{\tilde{q}+1}{\sqrt{\tilde{q}}} & 0 \\ 0 & 0 & \tilde{q}^l \\ 0 & -i\tilde{q}^{-2l-1} & 0 \\ 0 & 0 & \tilde{q}^{-l} \\ 0 & 0 & 0 \end{pmatrix}. \tag{C.8}$$

In components this reads

$$\bar{Y} = \sum_{a,b,c} y_a^{b,c} (z\tilde{q})^{la} z^{-lb} (z\tilde{q}^2)^{-lc}, \tag{C.9}$$

$$\bar{Y}^T = \sum_{a,b,c} \tilde{y}_{a,b}^{c} (z\tilde{q}^2)^{la} z^{lb} (z\tilde{q})^{-lc}. \tag{C.10}$$

The analogous operators to Y and Y^T in the loop basis we call M and $\tilde{M}$. These operators can be written as a combination of triangular plaquettes Fig. C.3.

$$M_j = \sum_{i=1}^{4} \mu_j^{(i)} m_i, \quad \tilde{M}_j = \sum_{i=1}^{4} \tilde{\mu}_j^{(i)} \tilde{m}_i \tag{C.11}$$

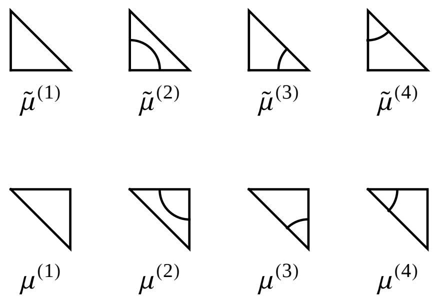


FIGURE C.3 – Entries of the operators M and $\tilde{M}$.

Now we equip the loops in μ_i and $\tilde{\mu}_i$ with orientations and multiply the weights m_3, m_4, $\tilde{m}_3$, $\tilde{m}_4$ by $\tilde{q}^l$ if the orientation is clockwise and by $\tilde{q}^{-l}$ if the orientation is counterclockwise. The weights m_2 and $\tilde{m}_2$ must be multiplied by a factor of $-i\tilde{q}^{-2l-1}$ and $i\tilde{q}^{2l+1}$ for the clockwise and counterclockwise orientation respectively. The weights of the M-operators of the loop basis read

$$m_1 = \tilde{q}^{1/2} + \tilde{q}^{-1/2} = \tilde{n}, \quad m_2 = m_3 = m_4 = 1, \tag{C.12}$$

$$\tilde{m}_1 = \tilde{q}^{1/2} + \tilde{q}^{-1/2} = \tilde{n}, \quad \tilde{m}_2 = \tilde{m}_3 = \tilde{m}_4 = 1. \tag{C.13}$$

The existence of this factorisation property gives rise, in particular, to a recurrence relation for the eigenstates. This recurrence relation was crucial in the computation of the ground state elements of the dilute $O(n = 1)$ Tempreley-Lieb model [48, 54, 57].

Appendix D

Résumé en français

D.1 Introduction

Cette thèse est consacrée à l'étude d'un modèle intégrable de vertex de Yang–Baxter, plus précisément, le modèle à dix-neuf vertex d'Izergin et Korepin (IK). Ce modèle est intéressant du point de vue de la physique statistique, la physique quantique en basse dimension, la théorie de la représentation des groupes quantiques, combinatoire, holomorphicité discrète, la géométrie algébrique, etc. Ce modèle est important parce que nous pouvons l'utiliser pour tester et développer des méthodes telles que l'Ansatz de Bethe, les équations q-Knizhnik–Zamolodchikov (qKZ), l'approche des opérateurs de vertex, opérateurs Q et les relations fonctionnelles. Ces méthodes sont très efficaces pour résoudre les problèmes dans le modèle à six vertex, qui est l'un des plus célèbres modèles intégrables de Yang–Baxter. Par conséquent, le modèle à six vertex devient un exemple très important qui nous permet d'apprendre les méthodes d'intégrabilité. D'autre part cette thèse peut être comprise comme une travail vers la généralisation des résultats qui ont été obtenus pour le modèle à six vertex.

Nous introduisons le modèle IK en utilisant la formulation de matrice de transfert. D'abord, nous introduisons les matrices R et $\check{R}$. Le matrice R est un endomorphisme de l'espace $V \otimes V$ où $V = \mathbb{C}^3$. La forme explicite de R est la suivante

$$R(u) = \left[\begin{array}{ccc|ccc|ccc} x_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & x_2 & 0 & x_5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & x_3 & 0 & x_6 & 0 & x_7 & 0 & 0 \\ \hline 0 & y_5 & 0 & x_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & y_6 & 0 & x_4 & 0 & x_6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & x_2 & 0 & x_5 & 0 \\ \hline 0 & 0 & y_7 & 0 & y_6 & 0 & x_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & y_5 & 0 & x_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x_1 \end{array}\right], \tag{D.1.1}$$

où chaque poids x_i et y_i dépend du paramètre u, qui est appelé le paramètre spectral.

Les poids de Boltzmann sont

$$
\begin{aligned}
x_1(u) &= 2\sinh\left(\frac{u}{2}-2\eta\right)\cosh\left(\frac{u}{2}-3\eta\right), \\
x_2(u) &= 2\sinh\frac{u}{2}\cosh\left(\frac{u}{2}-3\eta\right), \\
x_3(u) &= 2\sinh\frac{u}{2}\cosh\left(\frac{u}{2}-\eta\right), \\
x_4(u) &= 2\sinh\frac{u}{2}\cosh\left(\frac{u}{2}-3\eta\right)-2\sinh 2\eta\cosh 3\eta, \\
x_5(u) &= 2\mathrm{e}^{-\frac{1}{2}u}\sinh 2\eta\cosh\left(\frac{u}{2}-3\eta\right), \\
y_5(u) &= -\mathrm{e}^{u}x_5(u), \\
x_6(u) &= 2\mathrm{e}^{-\frac{1}{2}u+2\eta}\sinh 2\eta\sinh\frac{u}{2}, \\
y_6(u) &= \mathrm{e}^{u-4\eta}x_6(u), \\
x_7(u) &= -2\mathrm{e}^{-\frac{1}{2}u}\sinh 2\eta[\cosh\left(\frac{u}{2}-3\eta\right)+\mathrm{e}^{\eta}\sinh\frac{u}{2}], \\
y_7(u) &= -2\mathrm{e}^{\frac{1}{2}u}\sinh 2\eta[\cosh\left(\frac{u}{2}-3\eta\right)-\mathrm{e}^{-\eta}\sinh\frac{u}{2}].
\end{aligned}
\tag{D.1.2}
$$

Le paramètre supplémentaire η est appelé le paramètre de croisement. La matrice P est définie comme la matrice qui permute l'espaces V et W dans $V \otimes W$, de sorte que la matrice $\check{R}$ est égale à PR. Les poids de R ci-dessus sont tels que l'équation de Yang–Baxter (YB) est satisfaite

$$\check{R}_{2,3}(\lambda-\mu)\check{R}_{1,2}(\lambda)\check{R}_{2,3}(\mu) = \check{R}_{1,2}(\mu)\check{R}_{2,3}(\lambda)\check{R}_{1,2}(\lambda-\mu), \tag{D.1.3}$$

où $\check{R}_{1,2}(x) = \check{R}(x) \otimes I$ et $\check{R}_{2,3}(x) = I \otimes \check{R}(x)$. Nous pouvons maintenant passer à la formulation de la matrice de transfert. Considérons une matrice R qui agit sur l'espace $V_0 \otimes V_i$ et l'appelons $R_{0,i}$, ou tout simplement R_i. La matrice de monodromie est définie comme

$$M(u) = R_1(u)R_2(u)\dots R_N(u), \tag{D.1.4}$$

où $N \in \mathbb{N}$ est la longueur du système. Considérant $M(u)$ comme une matrice dans l'espace V_0 nous pouvons écrire

$$M(u) = \begin{bmatrix} A_1(u) & B_1(u) & B_2(u) \\ C_1(u) & A_2(u) & B_3(u) \\ C_2(u) & C_3(u) & A_3(u) \end{bmatrix}, \tag{D.1.5}$$

où les éléments de la matrice sont des opérateurs agissant sur $\mathcal{H} = V_1 \otimes \cdots \otimes V_N$, appelé l'espace physique. Maintenant, en raison de l'équation de Yang–Baxter (D.1.3) nous avons

$$\check{R}(\lambda-\mu)M(\lambda)M(\mu) = M(\mu)M(\lambda)\check{R}(\lambda-\mu). \tag{D.1.6}$$

Lorsque les produits $M(\lambda)M(\mu)$ et $M(\mu)M(\lambda)$ doit être comprise comme produits tensoriels de deux matrices M écrites sous la forme (D.1.5). Par conséquent, pour chaque nombre complexe u les éléments $\{A_i(u), B_i(u), C_i(u)\}_{i=1,2,3}$ forment une algèbre dont les relations peuvent être obtenues en utilisant l'équation (D.1.6). Cette algèbre est appelée l'algèbre de Yang–Baxter.

La matrice de transfert du modèle IK est définie par

$$T(u) = \mathrm{Tr}_{V_0}\ \kappa M(u), \tag{D.1.7}$$

où la trace est prise sur l'espace V_0 et κ est une matrice diagonale dans l'espace V_0 qui est appelé le twist. En utilisant la matrice de transfert IK (D.1.7) on peut écrire le Hamiltonien IK

$$H = T^{-1}(u)\frac{dT(u)}{du}|_{u=0}. \tag{D.1.8}$$

Pour comprendre la physique des systèmes décrits par le modèle IK nous devons diagonaliser ce Hamiltonien (ou de manière équivalente, la matrice de transfert). Pour ce faire, nous utilisons l'Ansatz de Bethe algébrique (ABA), voir Chapitre 1. L'ABA fonctionne dans le cadre de la matrice de transfert. Cette méthode donne une représentation des états propres de la matrice de transfert. Dans cette représentation les états propres sont construits comme des produits d'opérateurs de l'algèbre de YB agissant sur un état spécial appelé l'état de référence. En outre, chaque état propre dépend d'un ensemble de nombres complexes ζ_1,..,ζ_m, où m est le nombre total de particules dans le système. Ces nombres sont appelés racines de Bethe. Les valeurs propres de la matrice de transfert sont également écrits en termes de racines de Bethe correspondantes. L'ABA donne les équations de Bethe pour chaque état propre, dont les solutions sont les racines de Bethe. Les équations de Bethe forment un système d'équations algébriques. Nous pouvons les résoudre analytiquement seulement dans des cas très spéciaux.

Donnons un aperçu de la suite de ce chapitre. Dans la Section D.2 nous présentons l'Ansatz de Bethe de Tarasov pour le modèle IK. Dans cet Ansatz de Bethe les états propres de la matrice de transfert sont écrits dans une forme récursive. Nous résolvons cette récursion qui nous permet d'écrire les états propres sous une forme de produit. Ce produit a des applications potentielles dans la construction des facteurs de forme et des fonctions de corrélation du modèle IK. Comme nous avons appris de la littérature sur le modèle de six vertex, la fonction de partition avec des conditions aux bords de domaine (domain wall partition function DWPF) est extrêmement important pour l'étude des fonctions de corrélation du modèle de six vertex. Il est crucial que DWPF peut être écrit comme un déterminant. Par conséquent, il est nécessaire d'identifier un objet similaire pour le modèle IK. Section D.3 est consacrée à ce problème. Dans cette section, nous proposons une généralisation directe de la DWPF du modèle de six vertex à modèle à dix-neuf vertex. Pour cette fonction de partition nous écrivons deux relations de récurrence qui la fixent complètement. Nous sommes en mesure de trouver une expression de déterminant pour cette fonction de partition pour une valeur particulière du paramètre η (ou $q^3 = -1$, où $q = -e^{-2\eta}$). Le cas $q^3 = -1$ représente un système dans un régime d'interaction particulier. Afin de comprendre les régimes plus généraux, il faut connaître les racines de Bethe à q générique. Ces racines de

Bethe sont les racines des valeurs propres d'un opérateur qui est appelé l'opérateur Q de Baxter. La matrice de transfert peut être exprimée en termes de cette opérateur Q. Pour comprendre comment construire cette opérateur Q nous devons étudier la théorie de la représentation de l'algèbre assosié au modèle IK. La construction des représentations irréductibles de toute dimension k est l'objet de la Section D.4. Nous utilisons ces représentations dans la Section D.5 pour calculer un opérateur Q. Cela va comme suit. On peut calculer les matrices R à partir de la théorie de la représentation, comme opérateurs sur $V_0 \otimes V_i$. Ceci est basé sur la formule de Khoroshkin–Tolstoï. En utilisant la même formule et en prenant pour V_0 une représentation de dimension k dans laquelle k est envoyé à ∞ (de certaines manières), nous construisons les opérateurs L. En construisant les matrices de monodromie comme dans (D.1.4) où R est remplacé par un L et en prenant la trace sur la représentation de dimension infinie, nous arrivons à une notion des opérateurs Q en théorie de la représentation. Ces opérateurs Q devraient être reliés aux opérateurs Q de Baxter. Cette connexion est inconnue actuellement. En outre, du fait que prendre la trace de certaines matrices L représente un problème technique difficile, ces opérateurs Q ne sont pas actuellement disponibles.

Les valeurs propres de l'opérateur Q de Baxter sont appelées les fonctions Q. Quand $q^3 = -1$, il est possible de trouver la fonction Q correspondant à l'état fondamental (conjecturellement). Nous considérons cela dans la Section D.6. Nous montrons comment résoudre l'équation de Bethe pour l'état fondamental, puis calculer la valeur propre de la matrice de transfert et les composantes de l'état fondamental pour les petits systèmes.

D.2 L'Ansatz algébrique de Bethe pour le modèle IK

L'ABA pour le modèle IK a été étudiée par Tarasov [116]. Dans cette section, nous allons donner les résultats de sa construction. Tout d'abord définissons l'état du pseudo-vide $|0\rangle$

$$C_i(v)|0\rangle = 0, \quad A_i(v)|0\rangle = \alpha_i(v)|0\rangle, \quad B_i(v)|0\rangle \neq 0, \tag{D.2.1}$$

où $\alpha_i(\zeta) = x_i^L(\zeta)$. Les équations ci-dessus fixent l'état $|0\rangle$. Cet état correspond à l'état avec zéro particules. Si $v_1,\ v_0$ et v_{-1} sont les vecteurs de base de V, alors cet état correspond à produit tensoriel $v_1 \otimes \cdots \otimes v_1$ avec L multiplicateurs v_1. Le nombre de particules est un bon nombre quantique, donc on peut décomposer le espace quantique $\mathcal{H}$ dans les sous-espaces $\mathcal{H}_N$ avec un nombre de particules N fixe. Dans le secteur avec zéro particules l'état $|0\rangle$ est l'état propre de la matrice de transfert. Avec cet état on peut construire les autres états propres de la matrice de transfert qui appartiennent à d'autres secteurs N. Ces états propres est obtenu par l'action de certains polynômes Φ_N de l'algèbre de YB sur l'état $|0\rangle$. Tarasov a montré que ces polynômes obéissent à la relation de récurrence suivante

$$\Phi_N(\zeta_1,..,\zeta_N) = B_1(\zeta_1)\Phi_{N-1}(\zeta_2,..,\zeta_N) + B_2(\zeta_1)\sum_{i>1} c_{1,i}(\zeta_1,..,\zeta_N)\Phi_{N-2}(\zeta_2,..,\hat{\zeta}_i,..,\zeta_N), \tag{D.2.2}$$

avec la condition initiale $\Phi_0 = 1$ et

$$c_{l,i} = -x_1(\zeta_i)^{N-i+1} \sum_{j>l}^{n} \frac{1}{y(\zeta_l - \zeta_j)} \prod_{k>l,k\neq j}^{n} Z(\zeta_k - \zeta_j), \tag{D.2.3}$$

$$z(\zeta) = \frac{x_1(\zeta)}{x_2(\zeta)}, \quad y(\zeta) = \frac{x_3(\zeta)}{y_6(\zeta)},$$

$$Z(\zeta_k - \zeta_j) = \begin{cases} z(\zeta_k - \zeta_j) & \mathrm{si} k > j, \\ z(\zeta_k - \zeta_j)\omega(\zeta_j - \zeta_k) & \mathrm{si} k < j. \end{cases}$$

La matrice de transfert (avec κ égal à l'identité) peut être représentée comme

$$T(\zeta) = A_1(\zeta) + A_2(\zeta) + A_3(\zeta). \tag{D.2.4}$$

La matrice de transfert $T(u)$ et l'états Φ_N sont construits de les éléments de l'algèbre YB, alors on peut utiliser les relations de commutation pour montrer que les états $\Psi_N = \Phi_N|0\rangle$ sont les états propres de $T(u)$. Ce dernier est possible en vertu de l'hypothèse que certains termes disparaissent. Il est équivalent à un ensemble de conditions sur les paramètres $\zeta_1, \ldots, \zeta_N$ qui apparaissent dans (D.2.2). Ces conditions sont appelées équations de Bethe

$$\left(\frac{x_1(\zeta_a)}{x_2(\zeta_a)}\right)^L = \prod_{b\neq a=1}^{N} \frac{z(\zeta_a - \zeta_b)}{z(\zeta_b - \zeta_a)} \omega(\zeta_b - \zeta_a), \quad a = 1,2,...,N, \tag{D.2.5}$$

et les valeurs propres correspondantes $\Lambda_N(\zeta)$ sont

$$\Lambda_N(\zeta) = x_1(\zeta)^L \prod_{a=1}^{N} z(\zeta_a - \zeta) + x_2(\zeta)^L \prod_{a=1}^{N} \frac{z(\zeta - \zeta_a)}{\omega(\zeta - \zeta_a)} + x_3(\zeta)^L \prod_{a=1}^{N} \frac{x_2(\zeta - \zeta_a)}{x_3(\zeta - \zeta_a)}. \tag{D.2.6}$$

Pour le calcul de fonctions de corrélation nous avons besoin de considérer les produits scalaires de l'état propre avec l'état dual [84, 78]. Pour ce faire, il est important d'avoir une bonne formule pour l'état propre Ψ_N de la matrice de transfert. Ci-dessous, nous proposons une formule pour Ψ_N (ou Φ_N) qui est écrit en termes d'un produit qui résout la récurrence (D.2.2).

Tout d'abord nous présentons les règles d'ordre normale pour les opérateurs de l'algèbre YB $\{A_i(\zeta_j), B_i(\zeta_j), C_i(\zeta_j)\}_{i=1,2,3}$, $j = 1, \ldots, N$ pour certains N. Un monôme d'éléments de l'algèbre de YB est dans l'ordre normale si les opérateurs A_i sont situés entre les opérateurs B_j et C_k, où B_j sont sur le côté gauche à l'opérateurs A_i. Deux opérateurs avec des paramètres ζ_i et ζ_j pour $i \neq j$ ne commutent pas, en général, à cause de cela nous choisissons d'ordonner les opérateurs $\{X_1(\zeta_i), X_2(\zeta_j), X_3(\zeta_k)\}$, où $X = A,\ B$ ou C, selon l'indice de ζ, c'est à dire que le produit $X_{n_1}(\zeta_i)X_{n_2}(\zeta_j)$ est dans l'ordre normale si $i < j$. En outre, le produit de deux opérateurs, dont les indices de ζ coïncident doit être mis à zéro. Avec ce choix de l'ordre normale nous pouvons écrire

$$|\Psi_N\rangle = \oint \frac{\mathrm{d}x}{x^{N+1}} :\ \exp\left(x^2 \sum_{1\leq i<j\leq N} c_{i,j} B_2(\zeta_i) + x \sum_{1\leq i\leq N} B_1(\zeta_i)\right) :\ |0\rangle, \tag{D.2.7}$$

ou

$$|\Psi_N\rangle = \oint \frac{\mathrm{d}x}{x^{N+1}} : \prod_{i=1}^{N} e^{x\mathcal{B}(\zeta_i;x)} : |0\rangle,$$
$$\mathcal{B}(\zeta_i;x) = xB_2(\zeta_i) \sum_{i<j\leq N} c_{i,j} + B_1(\zeta_i),$$

où l'exponentielle est comprise comme une série entière en x et les monômes entre " :" sont considérés dans l'ordre normale. Aux fins de calcul, nous aimerions aussi avoir une représentation qui ne comporte pas un ordre normal. Une telle représentation peut être dérivée à partir de (D.2.7) au prix de l'introduction d'une nouvelle algèbre. Cette algèbre se compose de l'élément de l'unité $\mathbb{I}$ et d'éléments f_i, où ($i = 1, \ldots, N$), ces éléments doivent obéir les propriétés suivantes

$$[f_i,f_j] = 0, \quad f_i^2 = 0. \tag{D.2.8}$$

À l'aide de cette algèbre nous obtenons

$$|\Psi_N(\zeta_1,..,\zeta_N)\rangle =_N \langle\tilde{0}| \prod_{i=1}^{N} \beta(\zeta_i|\zeta_{i+1},..,\zeta_N)|\tilde{0}\rangle_N \otimes |0\rangle, \tag{D.2.9}$$

$$\beta(\zeta_i|\zeta_{i+1},..,\zeta_N) = \mathbb{I} + B_1(\zeta_i)f_i + B_2(\zeta_i) \times \sum_{j>i} c_{i,j} f_j f_i, \quad (i = 1,..N), \tag{D.2.10}$$

où $|\tilde{0}\rangle_N$ et ${}_N\langle\tilde{0}|$ sont l'état vide et l'état vide dual, respectivement, d'une représentation de l'algèbre des opérateurs f. Ces état sont définis en exigeant

$${}_N\langle\tilde{0}| \prod_{i=1}^{n} f_{a_i} |\tilde{0}\rangle_N = \delta_{n,N}. \tag{D.2.11}$$

Pour prouver que (D.2.9) satisfait les relation de récurrence de Tarasov il faut prendre le produit (D.2.9), puis isoler le premier terme $\beta(\zeta_1|\zeta_2,..,\zeta_n)$, écrire explicitement les états comme dans (D.2.10) et, enfin, utiliser les propriétés de l'algèbre des opérateurs f_i et de l'état vide (D.2.11). Pour plus de détails voir Chapitre 1.

En utilisant cette représentation pour les états propres de $T(u)$ et la représentation analogue pour les états duals

$$\langle\bar{\Psi}_N| = \oint \frac{\mathrm{d}x}{x^{N+1}} \langle 0| : \prod_{i=1}^{N} e^{x\mathcal{C}(\zeta_i;x)} : ,$$
$$\mathcal{C}(\zeta_i;x) = xC_2(\zeta_i) \sum_{i<j\leq N} \tilde{c}_{i,j} + C_1(\zeta_i),$$

où les coefficients $\tilde{c}$ sont donnés par la même formule que les coefficients c mais avec les poids x_6 et y_6 échangé, nous pouvons écrire le produit scalaire (défini comme)

$$S_N(\mu_1,..,\mu_N;\zeta_1,..,\zeta_N) = \langle\bar{\Psi}_N(\mu_1,..,\mu_N)|\Psi_N(\zeta_1,..,\zeta_N)\rangle, \tag{D.2.12}$$

de la manière suivante

$$S_N = \oint \frac{\mathrm{d}x\mathrm{d}y}{x^{N+1}y^{N+1}} \langle 0| : \prod_{i=1}^{N} e^{x\mathcal{C}(\mu_i;x)} : : \prod_{i=1}^{N} e^{y\mathcal{B}(\zeta_i;y)} : |0\rangle.$$

Nous pouvons éviter l'utilisation de l'ordre normal en introduisant les éléments f_i de l'algèbre ci-dessus. Le calcul du produit scalaire peut alors être accompli avec l'aide de la formule de Baker–Campbell–Hausdorff grâce à la nilpotence des opérateurs f_i. Nous espérons que cette nouvelle approche nous permettra de surmonter les difficultés techniques qui se posent dans le calcul des produits scalaires.

D.3 La fonction de partition DWBC pour le modèle IK

La fonction de partition avec des conditions aux bords de domaine pour le modèle IK peut être définie comme suit. Les éléments de la matrice R ont une représentation en termes de dix-neuf vertex qui sont présentés dans Fig. D.1. Prenons un domaine

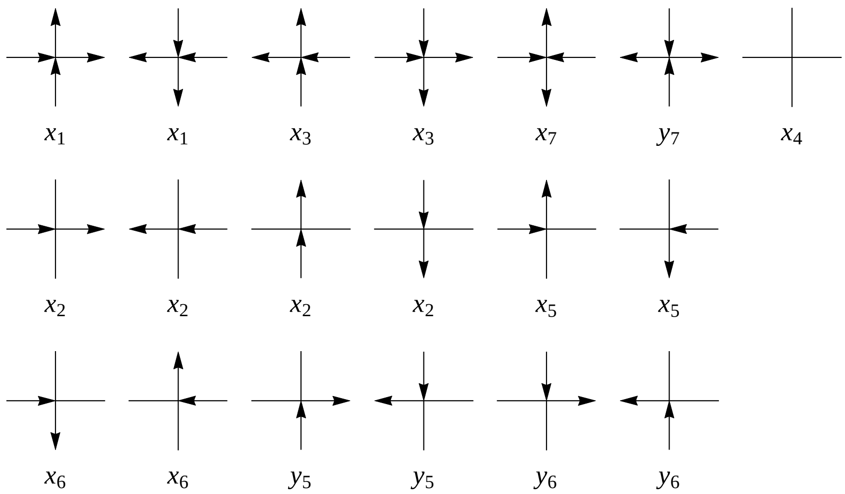


FIGURE D.1 – Les dix-neuf vertex et leurs poids.

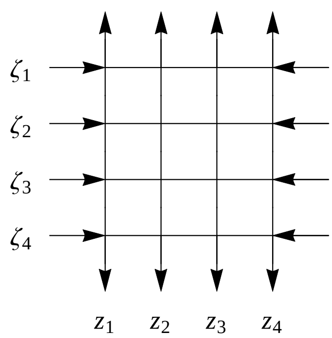


FIGURE D.2 – Les conditions aux bords de domaine sur un réseau 4×4. Les paramètres $\zeta_1, .., \zeta_4$ sont associés à des lignes horizontales, tandis que les paramètres $z_1, .., z_4$ sont associés à des lignes verticales.

carré D d'un réseau carré (voir par exemple Fig. D.2 ci-dessus) telle que les bords sont représentés par des arêtes. Pour chaque sommet avec coordonnées i et j, comptés

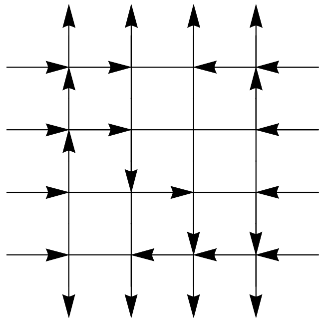


FIGURE D.3 – Une configuration typique du modèle IK avec les conditions aux bords de domaine sur un réseau 4×4.

(horizontalement et verticalement, respectivement) de 1 à N à partir du coin Nord-Ouest, on associe deux paramètres ζ_i et z_j. Ensuite, on choisit pour chaque sommet de ce réseau l'un des dix-neuf vertex de la Fig. D.1 de telle manière que les flèches ou leur absence sont d'accord sur chaque arête. De cette façon, nous produisons une configuration ε du modèle IK dans le domaine carré D. Le poids de cette configuration est égal au produit des poids de chacun des vertex qui font partie de la configuration ε. De plus, les poids sont des fonctions de ζ_i et z_j, où i, j sont les coordonnées des sommets correspondants. Notons $\Omega(D)$ l'ensemble de toutes les configurations ε du domaine D et $x_k(\zeta_i/z_j)$, $y_k(\zeta_i/z_j)$ les poids des dix-neuf vertex [1] par une fonction $w_{i,j}^{(\varepsilon)}$. Nous avons la fonction de partition

$$\mathcal{Z}(D) = \sum_{\varepsilon \in \Omega(D)} \prod_{1 \leq i,j \leq N} w_{i,j}^{(\varepsilon)}.$$

Sur l'ensemble des configuration $\Omega(D)$ choisissons seulement les éléments qui ont leurs arêtes aux bords comme sur la Fig. D.2. Ceci définit les conditions aux bords de domaine (DWBC) et la fonction de partition correspondante (maintenant désignée par Z ou Z_N pour le domaine $N \times N$) est appelée la fonction de partition du modèle IK avec DWBC. Maintenant, nous pouvons nous tourner vers la dérivation des relations de récurrence.

De la même manière que pour le modèle de six vertex nous dérivons les relations de récurrence en considérant les propriétés des poids. Plus précisément, choisissons un coin dans le domaine de la taille $N \times N$ avec DWBC et regardons sur toutes les configurations possibles dans ce coin (par example le coin avec les coordonnées (N, N)). Quand ζ_N est égal à z_N nous trouvons que toutes les configurations dans cette domain $N \times N$ se réduisent à la configuration du sous-domaine de la taille $(N-1) \times (N-1)$, tandis que les configurations à positions (i, N) et (N, i) (pour $i = 1 \ldots N$) sont gelées. Un phénomène similaire se produit lorsque nous fixons ζ_N égal à $-q^{-3} z_N$. En fait, en raison de l'équation de Yang–Baxter on peut montrer que la fonction de partition Z_N est symétrique dans les variables $z_1, .., z_N$ et dans les variables $\zeta_1, .., \zeta_N$. Par conséquent

1. Les poids $x_k(u)$, $y_k(u)$ sont considérés ici dans la convention multiplicatif $t = e^{u/2}$, $q = -e^{-2\eta}$, et t doit être remplecé par ζ_i/z_j pour tenir compte de la position i, j du sommet correspondant dans le domaine D.

la récurrence ci-dessus est valable si nous remplaçons ζ_N par un quelconque ζ_j et de même pour z_N. Le résultat de cette observation est les relations de récurrence suivantes

$$Z_N(\zeta_1,..,\zeta_j = z_i,..,\zeta_N|z_1,..,z_N) = F^N_{i,j} Z_{N-1}[\zeta_j,z_i], \tag{D.3.1}$$

$$Z_N(\zeta_1,..,\zeta_j = -q^{-3}z_i,..,\zeta_N|z_1,..,z_N) = G^N_{i,j} Z_{N-1}[\zeta_j,z_i], \tag{D.3.2}$$

où la fonction $Z_{N-1}[\zeta_j,z_i]$ ne dépend pas de ζ_j et z_i, et les fonctions $F^N_{i,j}$, et $G^N_{i,j}$ sont

$$F^N_{i,j} = (q^3+1)z_i \prod_{1\le k\ne i\le N}(q^2 z_i - z_k)(q^3 z_i + z_k) \prod_{1\le k\ne j\le N}(q^2\zeta_k - z_i)(q^3\zeta_k + z_i), \tag{D.3.3}$$

$$G^N_{i,j} = -q^{-N-1}(q^3+1)z_i \prod_{1\le k\ne i\le N}(z_i - q^2 z_k)(z_i + q^3 z_k) \prod_{1\le k\ne j\le N}(\zeta_k - z_i)(q\zeta_k + z_i). \tag{D.3.4}$$

Ces relations de récurrence avec le condition initiale $Z_0 = 1$ déterminent complètement un unique polynôme $Z_N(\zeta_1,..,\zeta_N,z_1,..,z_N)$. En utilisant (D.3.3) et (D.3.4) nous pouvons écrire Z_N utilisant l'interpolation de Lagrange [2]

$$\begin{aligned} Z_N(\zeta_1,..,\zeta_N|z_1,..,z_N) &= q^{3(N-1)} \sum_{k=1}^{N} Z_{N-1}[\zeta_N,z_k] \prod_{i=1,i\ne k}^{N} \frac{(\zeta_N - z_i)(\zeta_N + q^{-3}z_i)}{(z_k - z_i)} \\ &\times \Bigg((q^3\zeta_N + z_k)\prod_{i\ne k}(q^2 z_k - z_i) \prod_{1\le i\le N-1}(-z_k + q^2\zeta_i)(z_k + q^3\zeta_i) \\ &+ q^{2N-1}(\zeta_N - z_k)\prod_{i\ne k}(z_k - q^2 z_i) \prod_{1\le i\le N-1}(-z_k + \zeta_i)(z_k + q\zeta_i)\Bigg). \end{aligned} \tag{D.3.5}$$

Il est difficile de voir si la fonction de partition Z avec q générique peut être écrit comme un déterminant. Lorsque $q^3 = -1$ la récurrence devient plus simple

$$Z_N(\zeta_1,..,\zeta_j = z_i,..,\zeta_N|z_1,..,z_N) = P_{i,j} Z_{N-1}[\zeta_j,z_i], \tag{D.3.6}$$

où

$$\begin{aligned} &P(x|\zeta_1,..,\zeta_{N-1},z_1,..,z_{N-1}) = \\ &(-q)^N \left(q\prod_{i=1}^{N-1}(\zeta_i + qx)\prod_{i=1}^{N-1}(z_i + x/q) + \frac{1}{q}\prod_{i=1}^{N-1}(\zeta_i + x/q)\prod_{i=1}^{N-1}(z_i + qx)\right), \end{aligned} \tag{D.3.7}$$

la condition initiale devient $Z_1 = 1$ et nous trouvons une expression de déterminant

$$Z_N(\zeta_1,..,\zeta_N,z_1,..,z_N) = \det_{1\le i,j\le N-1} \Delta_{3j-i,N}(\zeta_1,..,\zeta_N,z_1,..,z_N), \tag{D.3.8}$$

Ce déterminant est écrit en termes de polynômes symétriques $\Delta_{i,n}$ qui sont générés par le polynôme $P_N(x)$

$$P_N(x) = P(x|\zeta_1,..,\zeta_N,z_1,..,z_N) = (-q)^N \sum_{i=0}^{2N} x^i \Delta_{2N-i,N}(\zeta_1,..,\zeta_N,z_1,..,z_N), \tag{D.3.9}$$

2. Il est facile de calculer le dénominateur de Z, donc nous pouvons toujours normaliser Z de sorte qu'il devient un polynôme dans les paramètres ζ et z.

Pour prouver (D.3.8) on doit appliquer une série de manipulations sur les colonnes et des lignes de la matrice $\Delta_{3j-i,N}$. Après cela on arrive à un déterminant de $\Delta_{3j-i,N-1}$ dans les premiers $N-1$ lignes et $N-1$ colonnes, la dernière ligne est zéro, sauf l'élément (N,N) qui est égal exactement au polynôme (D.3.7). Voir les détails dans Chapitre 1.

D.4 Représentations irréductibles de $U_q(A_2^{(2)})$

La connaissance de la représentation fondamentale (dimension égal à trois) de l'algèbre $U_q(A_2^{(2)})$ permet de calculer la matrice R du modèle IK. Nous pouvons faire ceci en utilisant la formule de Khoroshkin–Tolstoï (KT), dans laquelle on définit la matrice $\mathcal{R}$ universelle comme un élément de $U_q(\mathfrak{b}^+) \otimes U_q(\mathfrak{b}^-)$, où $U_q(\mathfrak{b}^+)$ et $U_q(\mathfrak{b}^-)$ sont les deux sous-algèbres de Borel de $U_q(A_2^{(2)})$. Par définition l'élément $\mathcal{R}$ satisfait

$$\begin{aligned}
&\Delta'(x) = \mathcal{R}\Delta(x)\mathcal{R}^{-1}, \quad \forall x \in \mathcal{A}, \\
&(\Delta \otimes \mathrm{id})\mathcal{R} = \mathcal{R}_{1,3}\mathcal{R}_{2,3}, \quad (\mathrm{id} \otimes \Delta)\mathcal{R} = \mathcal{R}_{1,3}\mathcal{R}_{1,2}.
\end{aligned}$$

La formule KT est une formule explicite pour la matrice $\mathcal{R}$ écrite en termes de générateurs de l'algèbre $U_q(A_2^{(2)})$. En prenant une représentation $V \otimes W$ de $U_q(A_2^{(2)}) \otimes U_q(A_2^{(2)})$ pour $\mathcal{R}$, par les équations ci-dessus, il est garanti que le résultat satisfera l'équation de Yang–Baxter. La matrice R de IK est la représentation de $\mathcal{R}$ sur l'espace $V \otimes V$ où V est la représentation fondamentale de $U_q(A_2^{(2)})$. En suivant les exemples donnés en [5, 4], nous aimerions trouver des représentations de dimension infinie $V^{(\infty)}$ de $U_q(\mathfrak{b}^+)$ (ou $U_q(\mathfrak{b}^-)$) qui peuvent être utilisés pour la construction de l'opérateur Q. Ces derniers opérateurs sont les matrices de transfert construits comme dans (D.1.7) où la trace est prise sur une représentation de dimension infinie $V^{(\infty)}$ de $U_q(\mathfrak{b}^+)$. À cet effet, nous trouvons certains représentations irréductibles de plus haut poids $V^{(k)}$ (modules de Kirillov–Reshetikhin) de dimension $(k+1)(k+2)/2$, où $k \in \mathbb{N}$. Nos formules dépendent explicitement de k, prendre la limite $k \to \infty$ conduit à des représentations de dimension infinie $V^{(\infty)}$. Selon le choix du vecteur de référence dans $V^{(k)}$ on arrive à différentes représentations $V^{(\infty)}$. Soyons plus spécifiques. L'algèbre $U_q(A_2^{(2)})$ est associée à la matrice de Cartan

$$C^s = \begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix},$$

et est définie par les générateurs (de Drinfeld) $x_r^{\pm}$, $h_{\pm m}$ et K ($r \in \mathbb{Z}$, $m \in \mathbb{Z}_+$) qui satisfont les relations

$$\begin{aligned}
&KK^{-1} = K^{-1}K = 1, \quad Kh_k = h_kK, \quad h_kh_l = h_lh_k, \\
&Kx_k^{\pm}K^{-1} = q^{\pm 1}x_k^{\pm}, && \text{(D.4.1)} \\
&[x_r^+, x_s^-] = \frac{\psi_{r+s}^+ - \psi_{r+s}^-}{q - q^{-1}}, && \text{(D.4.2)} \\
&[h_r, x_s^{\pm}] = \pm\frac{[r]}{r}(q^r + q^{-r} + (-1)^{r+1})x_{r+s}^{\pm}, && \text{(D.4.3)}
\end{aligned}$$

$$
\begin{aligned}
&x^{\pm}_{r+2}x^{\pm}_{s} + (q^{\mp 1} - q^{\pm 2})x^{\pm}_{r+1}x^{\pm}_{s+1} - q^{\pm 1}x^{\pm}_{r}x^{\pm}_{s+2} \\
&= q^{\pm 1}x^{\pm}_{s}x^{\pm}_{r+2} + (q^{\pm 2} - q^{\mp 1})x^{\pm}_{s+1}x^{\pm}_{r+1} - q^{\pm 1}x^{\pm}_{s+2}x^{\pm}_{r},
\end{aligned}
\tag{D.4.4}
$$

$$
\mathrm{Sym}(q^{3/2}x^{\pm}_{r\mp 1}x^{\pm}_{s}x^{\pm}_{t} - (q^{1/2}+q^{-1/2})x^{\pm}_{r}x^{\pm}_{s\mp 1}x^{\pm}_{t} + q^{-3/2}x^{\pm}_{r}x^{\pm}_{s}x^{\pm}_{t\mp 1}) = 0, \tag{D.4.5}
$$

$$
\mathrm{Sym}(q^{-3/2}x^{\pm}_{r\pm 1}x^{\pm}_{s}x^{\pm}_{t} - (q^{1/2}+q^{-1/2})x^{\pm}_{r}x^{\pm}_{s\pm 1}x^{\pm}_{t} + q^{3/2}x^{\pm}_{r}x^{\pm}_{s}x^{\pm}_{t\pm 1}) = 0, \tag{D.4.6}
$$

où Sym désigne une somme sur toutes les permutations de r, s et t. Les éléments $\psi^{\pm}_{k}$ apparaissant dans l'équation (D.4.2) peuvent être écrits en termes de générateurs h_l en utilisant la relation

$$
\Psi^{\pm}(u) = \sum_{k=0}^{\infty} \psi^{\pm}_{\pm k}u^{\pm k} = K^{\pm 1}\exp\left(\pm(q-q^{-1})\sum_{l=1}^{\infty} h_{\pm l}u^{l}\right). \tag{D.4.7}
$$

Soit V une représentation de $U_q(A_2^{(2)})$. Un vecteur $v \in V$ est appelé un vecteur de plus haut poids dans le sens de la présentation de Drinfeld si il satisfait

$$
x^{+}_{k}v = 0, \qquad \psi^{\pm}_{k}v = \psi^{\pm}_{k;0}v, \tag{D.4.8}
$$

pour $i \in I$, $k \in \mathbb{Z}$ et certains nombres complexes $\psi^{\pm}_{i,k;0}$. Cette représentation est une représentation de plus haut poids si pour un vecteur de plus haut poids v nous avons $V = U_q(A_2^{(2)}).v$. L'ensemble des nombres $\{\psi^{\pm}_{i,k;0}\}$ est appelé le plus haut poids de la représentation V. D'après le théorème de classification (Théorème 3.3 de la [25]) il résulte que les représentations irréductibles de dimension finie des algèbres quantiques affines peuvent être définies par les polynômes de Drinfeld $P(u)$. Les coefficients de ces polynômes sont précisément les nombres $\psi^{\pm}_{i,k;0}$ apparaissant dans (D.4.8). Nous avons

$$
\sum_{r=0}^{\infty} \psi^{+}_{r}u^{r}.v = q^{\deg(P)}\frac{P(q_i^{-2}u)}{P(u)}v = \sum_{r=0}^{\infty} \psi^{-}_{r}u^{-r}.v,
$$

au sens que les termes de gauche et de droite sont les développements de Laurent du terme du milieu autour de 0 et ∞, respectivement. La classe de représentations qui est pertinente pour nous, appelé les modules de Kirillov–Reshetikhin, est définie par le polynôme de Drinfeld

$$
P(z) = \prod_{l=1}^{k}(1 - q^{k-2l+1}z).
$$

Cette représentation est marqués par le nombre k et, d'après Proposition 10.1 de [61], la dimension de cette représentation est égal à $(k+1)(k+2)/2$. Nous représentons l'espace $V^{(k)}$ comme $V = \bigoplus_{0\le n_1\le n_2\le k} v_{n_1,n_2}$, et l'élément de Cartan K définit la décomposition de $V^{(k)}$ en les espaces de poids

$$
V = \bigoplus_{p=0}^{2k} V_p, \qquad V_p = \{v_{n_1,n_2} \in V | Kv_{n_1,n_2} = q^{k-p}v_{n_1,n_2},\, p = n_1 + n_2\}. \tag{D.4.9}
$$

Les éléments de matrice de $\psi_k^{\pm}$ dans $V^{(k)}$ peuvent également être extraits de la Proposition 10.1 de [61]. Nous pouvons écrire

$$\sum_{k=0}^{\infty} u^{\pm k} \psi^{\pm}_{k\ n_1,n_2} = q^{\deg(P_{n_1,n_2})-\deg(Q_{n_1,n_2})} \frac{P_{n_1,n_2}(q^{-1}u)}{P_{n_1,n_2}(qu)} \frac{Q_{n_1,n_2}(qu)}{Q_{n_1,n_2}(q^{-1}u)} = \Psi^{\pm}_{n_1,n_2}(u), \tag{D.4.10}$$

où

$$P_{n_1,n_2}(u) = \prod_{j=1}^{k-n_2} (1 - auq^{2(j-1)}) \prod_{j=k-n2+1}^{k-n_1} (1 + auq^{2j-1}), \tag{D.4.11}$$

$$Q_{n_1,n_2}(u) = \prod_{j=k-n_2+1}^{k-n_1} (1 - auq^{2j}) \prod_{j=k-n1+1}^{k} (1 + auq^{2j+1}). \tag{D.4.12}$$

Donc nous connaissons la forme explicite de $\psi_k^{\pm}$ et, par conséquent, en utilisant la formule (D.4.7), on peut calculer les éléments de matrice $h_{\pm l}$. En remplaçant ces derniers dans les équations (D.4.3) dans la représentation $V^{(k)}$ on peut voir que l'action des opérateurs $x_r^{\pm}$ est de la forme

$$x_r^{\pm} v_{n_1,n_2} = \alpha^{\pm}_{n_1,n_2}(r) v_{n_1\mp1,n_2} + \beta^{\pm}_{n_1,n_2}(r) v_{n_1,n_2\mp1}, \tag{D.4.13}$$

où $\alpha^{\pm}_{n_1,n_2}(r)$ et $\beta^{\pm}_{n_1,n_2}(r)$ sont les éléments de matrices correspondants. Ce dernier résultat signifie que la représentation sur l'espace $V^{(k)}$ peut être représenté par un graphe dont les noeuds sont les vecteurs v_{n_1,n_2} et les arêtes relient les vecteurs qui sont liés par l'action d'un opérateur $x_r^{\pm}$ (voir Fig. D.4). Ensuite, nous calculons les éléments

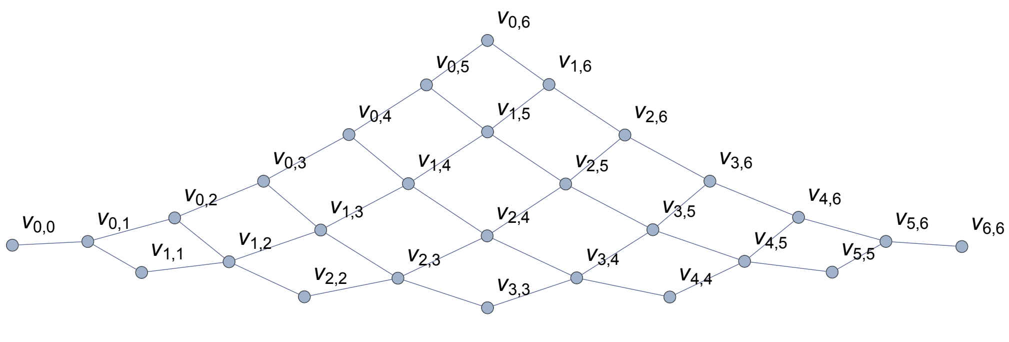


FIGURE D.4 – Le graphe de l'espace vectoriel $V^{(6)}$.

des matrices $\alpha^{\pm}_{n_1,n_2}(r)$ et $\beta^{\pm}_{n_1,n_2}(r)$. En utilisant les relations de commutation entre les générateurs de Drinfeld, on obtient un certain nombre de relations de récurrence pour l'éléments α et β. Après la résolution de ces relations de récurrence, nous obtenons

$$\begin{aligned} x_m^+ v_{i,j} &= A(-1)^m q^{2m-2im} v_{i-1,j} \\ &+ B^{-1} \frac{\left(q^{2j+1}+1\right)\left(q^{2j}-q^{2i}\right)\left(q^{2k+2}-q^{2j}\right) q^{i+(1-2j)m-j-k+1}}{(q-1)^2(q+1)\left(q^{2i+1}+q^{2j}\right)\left(q^{2i}+q^{2j+1}\right)} v_{i,j-1}, \quad \text{(D.4.14)} \\ x_m^- v_{i,j} &= A^{-1} \frac{(-1)^m \left(q^{2i+2}-1\right)\left(q^{2j}-q^{2i}\right)\left(q^{2i+1}+q^{2k+2}\right) q^{-2im-i+j-k}}{(q-1)^2(q+1)\left(q^{2i+1}+q^{2j}\right)\left(q^{2i}+q^{2j+1}\right)} v_{i+1,j} \\ &+ B q^{-(2j+1)m} v_{i,j+1}, \quad \text{(D.4.15)} \end{aligned}$$

où A et B sont des paramètres libres. Ces formules dépendent explicitement de k, d'où la limite $k \to \infty$ nous donne les représentations de dimension infinie correspondant au vecteur de référence $v_{0,0}$. Choisir d'autres vecteurs de référence (voir par example Section 4.6, Eq. (4.6.4)) nous amène à "différentes" représentations de dimension infinie. Dans la Section D.5, nous utilisons ces représentations pour calculer les matrices L. Ces dernières matrices jouent le même rôle dans la construction de l'opérateur Q que la matrice R pour la construction de la matrice de transfert.

D.5 Matrice $\mathcal{R}$ universelle pour $U_q(A_2^{(2)})$

La matrice universel $\mathcal{R}$ est définie par les équations suivantes

$$\Delta'(x) = \mathcal{R}\Delta(x)\mathcal{R}^{-1}, \quad \forall x \in \mathcal{A}, \quad \text{(D.5.1)}$$

$$(\Delta \otimes \mathrm{id})\mathcal{R} = \mathcal{R}_{1,3}\mathcal{R}_{2,3}, \quad (\mathrm{id} \otimes \Delta)\mathcal{R} = \mathcal{R}_{1,3}\mathcal{R}_{1,2}. \quad \text{(D.5.2)}$$

Prenons l'Ansatz

$$\mathcal{R} = \sum_i u_i \otimes v_i,$$

où u_i et v_i sont les éléments de (certaines extensions de) $U_q(\mathfrak{b}^+)$ et $U_q(\mathfrak{b}^-)$ respectivement. Dans ce Ansatz la solution de (D.5.1) et (D.5.2) est un produit d'exponentielles sur l'ensemble des racines plus élevées de l'algèbre $U_q(A_2^{(2)})$. Nous écrivons les racines simples α_0 et α_1 comme

$$\alpha = \alpha_1, \quad \alpha_0 = \delta - 2\alpha,$$

puis les racines positifs sont

$$\begin{aligned} \Delta_+ &= \{\alpha + m\delta | m \in \mathbb{Z}_{\geq 0}\} \cup \{2\alpha + (2m+1)\delta | m \in \mathbb{Z}_{\geq 0}\} \\ &\cup \{m\delta | m \in \mathbb{Z}_{>0}\} \cup \{\delta - 2\alpha + 2m\delta | m \in \mathbb{Z}_{\geq 0}\} \cup \{\delta - \alpha + m\delta | m \in \mathbb{Z}_{\geq 0}\} \quad \text{(D.5.3)} \end{aligned}$$

Ensuite, nous choisissons l'ordre dans l'ensemble des racines

$$(\delta - \gamma) + l\delta \prec m\delta \prec \gamma + k\delta, \qquad \gamma = \alpha, 2\alpha, \quad k, l \in \mathbb{Z}_{\geq 0},\ m \in \mathbb{Z}_{>0}. \quad \text{(D.5.4)}$$

La formule KT est écrite comme un produit de quatre termes

$$\begin{aligned}
&\mathcal{R} = \mathcal{R}_{\succ\delta}\mathcal{R}_{\delta}\mathcal{R}_{\prec\delta}\mathcal{K}, &\text{(D.5.5)}\\
&\mathcal{R}_{\succ\delta} = \prod_m \mathcal{R}_{\delta-\alpha,m}\prod_m \mathcal{R}_{\delta-2\alpha,m},\\
&\mathcal{R}_{\delta-\gamma,m} = \exp_{q_\gamma}((q-q^{-1})e_{\delta-\gamma+m\delta}\otimes f_{\delta-\gamma+m\delta}), \quad \gamma=\alpha,\, 2\alpha,\\
&\mathcal{R}_{\delta} = \exp\left((q^{1/2}-q^{-1/2})\sum_{m>0}\frac{m}{[2m]_{q^{1/2}}(q^m+q^{-m}+(-1)^{m+1})}\; e_{m\delta}\otimes f_{m\delta}\right),\\
&\mathcal{R}_{\prec\delta} = \prod_m \mathcal{R}_{\alpha,m}\prod_m \mathcal{R}_{2\alpha,m},\\
&\mathcal{R}_{\gamma,m} = \exp_{q_\gamma}((q-q^{-1})e_{\gamma+m\delta}\otimes f_{\gamma+m\delta}), \quad \gamma=\alpha,\, 2\alpha,
\end{aligned}$$

où la q-exponentielle est la série

$$\begin{aligned}
&\exp_q(x) = 1 + x + \frac{x^2}{(2)_q!} + \frac{x^3}{(3)_q!} + .., &\text{(D.5.6)}\\
&(n)_q! = (n)_q(n-1)_q..(2)_q(1)_q, \qquad (n)_q = \frac{q^n-1}{q-1}.
\end{aligned}$$

Le dernier terme $\mathcal{K}$ est défini comme suit. Pour deux quelconques vecteurs $v \in V^{(m)}$ et $w \in V^{(l)}$, nous avons $kv = q^{(\lambda,\alpha)}v$ et $kw = q^{(\mu,\alpha)}w$, où λ et μ sont les poids de v et w, alors

$$\mathcal{K}v\otimes w = q^{(\lambda,\mu)}v\otimes w. \tag{D.5.7}$$

La formule KT est écrite en termes de générateurs de Drinfeld–Jimbo ; dans la Section 4.2, nous montrons comment réécrire la formule KT en termes de générateurs de Drinfeld. Cela se révèle être très pratique du point de vue technique ; d'ailleurs, les représentations que nous avons construit avant peuvent être substitués dans la formule KT écrit dans la présentation de Drinfeld.

Il est possible d'évaluer analytiquement la matrice $\mathcal{R}$ universelle sur l'espace $V\otimes W$ si l'un des facteurs V ou W est de dimension basse. Plus précisément, si $W = V = V^{(1)}$, deux représentations fondamentales, la formule KT donne la matrice R de modèle IK. La deuxième application de la formule KT est l'évaluation de $\mathcal{R}$ sur l'espace $V^{(1)}\otimes V^{(k)}$, dénotée $R^{(k)}$. Avec la matrice $R^{(k)}$ on peut calculer les matrices de transfert avec des espaces auxiliaires de dimension plus élevée. Ces matrices de transfert sont importantes dans la théorie des relations fonctionnelles.

Rappelons que la représentation $V^{(k)}$ de la section précédente est obtenue avec le vecteur de référence $v_{0,0}$ (voir Fig. D.4). On peut modifier $V^{(k)}$ en changeant son état de référence (voir par example Section 4.6, Eq. (4.6.4)). Envoyer k à ∞ dans $R^{(k)}$ nous donne trois matrice L. Ces matrices L correspondent aux trois choix de vecteurs de référence : $v_{0,0}$, $v_{0,k}$ et $v_{k,k}$ (voir [62] pour les explications). Malheureusement, le calcul des trois opérateurs Q se révèle être très difficile techniquement. L'opérateur Q correspondant à $v_{0,k}$ est le plus simple d'entre eux. Cela est dû au fait q'il peut être écrit en termes d'une algèbre de q-bosons (qOsc)

$$L = \begin{pmatrix} \frac{\lambda q^8}{\kappa} + \zeta\kappa\lambda q^5 & q^7 c_1^\dagger + q^4 c_2^\dagger\left(\zeta\kappa^2 + q^3\right) & c_1^{\dagger 2}\frac{\kappa q^7}{\lambda(q+1)} + c_1^\dagger c_2^\dagger \frac{\kappa q^6}{\lambda} + c_2^{\dagger 2}\frac{\kappa q^4\left(\zeta\kappa^2+q^3\right)}{\lambda(q+1)} \\ -q^4 c_1 \zeta\lambda^2 & -q^2 c_2^\dagger c_1 \zeta\kappa\lambda + q^4(q^4-\zeta) & c_1^\dagger \frac{\kappa q^7}{\lambda} + c_2^\dagger \frac{\kappa q^3\left(q^4-\zeta\right)}{\lambda} - c_2^{\dagger 2} c_1 \frac{\zeta\kappa^2 q}{q+1} \\ c_1^2 \frac{\zeta\lambda^3 q^5}{\kappa(q+1)} & c_2^\dagger c_1^2 \frac{\zeta\lambda^2 q^2}{q+1} + c_1 \frac{\zeta\lambda q^4}{\kappa} & q^2 c_2^\dagger c_1 \zeta + c_2^{\dagger 2} c_1^2 \frac{\zeta\kappa\lambda}{(q+1)^2} + \frac{q^3\left(\zeta+\kappa^2 q^5\right)}{\kappa\lambda} \end{pmatrix},$$

où les éléments $c_1,\ c_1^\dagger,\ c_2,\ c_2^\dagger$ et $\kappa,\ \lambda$ forment une algèbre q-Osc avec les relations de commutation

$$\begin{aligned}
&[c_1,c_2^\dagger] = [c_2,c_1^\dagger] = [\kappa,\lambda] = 0, \\
&\kappa c_1 = q^{-1} c_1 \kappa, \quad \kappa c_1^\dagger = q c_1^\dagger \kappa, \\
&\kappa c_2 = \kappa c_2, \quad \kappa c_2^\dagger = \kappa c_2^\dagger, \\
&\lambda c_1 = c_1 \lambda, \quad \lambda c_1^\dagger = c_1^\dagger \lambda, \\
&\lambda c_2 = q\lambda c_2, \quad \lambda c_2^\dagger = q^{-1}\lambda c_2^\dagger, \\
&c_1 c_2 - c_2 c_1\ q^{-1} = 0, \\
&c_2^\dagger c_1^\dagger - c_1^\dagger c_2^\dagger\ q^{-1} = 0, \\
&c_1 c_1^\dagger = \frac{(q+1)\left(1-\kappa^2 q^2\right)}{\kappa\lambda}, \quad c_1^\dagger c_1 = -\frac{\left(\kappa^2-1\right)q(q+1)}{\kappa\lambda}, \\
&c_2 c_2^\dagger = -\frac{\left(\lambda^2-1\right)q(q+1)}{\kappa\lambda}, \quad c_2^\dagger c_2 = \frac{(q+1)\left(1-\lambda^2 q^2\right)}{\kappa\lambda}.
\end{aligned}$$

Afin de comprendre la relation entre les opérateurs Q venant de la théorie de la représentation et les opérateurs Q de Baxter il faut régler d'abord le problème du calcul des deux opérateurs Q restants.

D.6 L'état fondamental du modèle IK au régime $q^3 = -1$

Nous avons vu que la fonction de partition DWBC a une représentation de déterminant à $q^3 = -1$. Il se trouve que dans ce cas spécial $q^3 = -1$ il est possible de calculer d'autres quantités intéressantes. Notamment, nous pouvons trouver certaines solutions des équations de Bethe. Avec la connaissance des racines de Bethe nous pouvons calculer l'état propre (et la valeur propre) correspondant, en utilisant l'Ansatz algébrique de Bethe présenté auparavant. En fait, on conjecture que c'est l'état fondamental. Cette conjecture est basée sur de calculs pour les systèmes de petites tailles.

Dans cette section, nous allons nous limiter à des états propres de secteur de N particules et la longueur du système est également pris égal à N. En outre, nous considérons les conditions aux bords tordues avec la twist $\kappa = \mathrm{diag}\{q, 1, 1/q\}$. Les équations de Bethe correspondant sont

$$\prod_{i=1}^{N} \frac{\left(z_i^2 - q^2\zeta_j^2\right)}{q\left(z_i^2 - \zeta_j^2\right)} - q^{-1} \prod_{\substack{i=1 \\ i\neq j}}^{N} \frac{\left(\zeta_i^2 - q^2\zeta_j^2\right)\left(q\zeta_i^2 + \zeta_j^2\right)}{\left(q^2\zeta_i^2 - \zeta_j^2\right)\left(\zeta_i^2 + q\zeta_j^2\right)} = 0, \quad \text{for } j = 1\ldots N. \tag{D.6.1}$$

Les paramètres z_i sont les inhomogénéités. Tout d'abord, nous remarquons que $\zeta_j = 0$ résout cette équation. Cela est une conséquence du choix spécial du twist. Ensuite, nous notons que cette équation dépend des carrés des racines de Bethe, cela signifie que changer le signe de l'une des racines de Bethe n'a aucun effet et donne ainsi une solution séparée. Introduisons de nouvelles variables $\lambda_i = \zeta_i^2$ et $\lambda = \zeta^2$. Les équations de Bethe peuvent être écrites comme

$$\frac{F'\left(q^2\lambda_j\right)}{F'\left(\lambda_j\right)} + \frac{Q'\left(-q^{-1}\lambda_j\right)Q'\left(q^2\lambda_j\right)}{qQ'\left(q^{-2}\lambda_j\right)Q'\left(-q\lambda_j\right)} = 0, \tag{D.6.2}$$

où

$$Q'(\lambda) = \prod_{i=1}^{N}\left(\lambda - \lambda_i\right), \qquad F'(\lambda) = \prod_{i=1}^{N}\left(\lambda - z_i^2\right). \tag{D.6.3}$$

Soit $\omega = e^{i\pi/3}$ et mettez $q = \omega$. Il peut être directement vérifiée que

$$Q'(t) = \mathrm{const}\sqrt{t}\left(\tilde{F}\left(-\sqrt{t}\right)\tilde{F}\left(-\omega\sqrt{t}\right) - \tilde{F}\left(\sqrt{t}\right)\tilde{F}\left(\omega\sqrt{t}\right)\right), \tag{D.6.4}$$

où

$$\tilde{F}(x) = \prod_{i=1}^{N}\left(x - z_i\right), \tag{D.6.5}$$

résout l'équation (D.6.2). Ça veut dire qu'il existe un état propre (et aussi un état propre dual) qui correspond à cette solution des équations de Bethe. Nous appelons cet état ψ_N et conjecturons que cet état est l'état fondamental. De (D.6.4) et (D.6.3) nous trouvons que les polynômes symétriques élémentaires dans les racines de Bethe, notée $E_i^{(\lambda)}$, peuvent être écrits explicitement en termes de polynômes symétriques élémentaires dans les inhomogénéités, notée $E_i^{(z)}$

$$E_m^{(\lambda)} = -\frac{(-1)^m}{2E_1^{(z)}\left(1-\omega^2\right)}\sum_{i=0}^{2m+2} E_i^{(z)}E_{2m-i+1}^{(z)}(\omega^{4m+2}\omega^i - \omega^{-i}). \tag{D.6.6}$$

Notant en outre que les expressions pour les états propres et les valeurs propres données par l'Ansatz algébrique de Bethe sont symétriques dans les racines de Bethe, nous pouvons utiliser (D.6.6) pour obtenir la solution explicite. La valeur propre pour les systèmes de taille générale N est

$$\Lambda(t) = \frac{\omega^2\left(\tilde{F}\left(-t\omega\right)\tilde{F}\left(-t\omega^2\right) + \tilde{F}\left(t\omega\right)\tilde{F}\left(t\omega^2\right)\right)}{\tilde{F}\left(-t\omega\right)\tilde{F}\left(t\omega\right)}. \tag{D.6.7}$$

L'état propre ψ_N, cependant, ne peut pas être écrit sous une forme fermée pour N grand. Les exemples de ψ_N pour petits N sont présentés dans le Chapitre 5. Dans le même chapitre, on peut aussi trouver les produits scalaires de l'état ψ_N et son état dual pour $N = 2$.

On peut vérifier que l'expression (D.2.6) pour les valeurs propres de $T(u)$ et les équations de Bethe dépendent des carrés des inhomogénéités et peuvent être rédigés en termes de Q'. Les fonctions Q', à son tour, dépendent à la fois des puissances paires et impaires des inhomogénéités. C'est-à-dire que changer le signe d'un sous-ensemble des inhomogénéités $z_1, .., z_N$ conduit également à une solution des équations de Bethe. Cette solution, en raison de (D.2.6), donne automatiquement une autre valeur propre de la matrice de transfert. En inspectant la formule (D.6.6), nous voyons que de cette manière nous pouvons produire 2^{N-1} différentes valeurs propres de la matrice de transfert. Les états propres correspondants pour les systèmes de petites tailles peuvent également être calculées. Ces calculs sont limités à un certain sous-ensemble des états propres du secteur avec des N particules dans le régime quand q est une racine de l'unité. Résolvant les équations de Bethe pour les cas de q générique n'est pas possible analytiquement. Toutefois, certains outils permettent d'obtenir quelques informations sur les racines de Bethe. L'un de ces outils est d'utiliser l'opèrateur Q. Pour le construire il faut d'abord étudier la théorie de la représentation du groupe quantique qui sous-tend le modèle IK. Ce groupe quantique (l'algèbre de Lie quantique affine) est appelée $U_q(A_2^{(2)})$, et nous nous tournons vers l'étude de certains des aspects de la théorie de la représentation dans la section suivante.

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